Available online at www.sciencedirect.com
ScienceDirect
Stochastic Processes and their Applications 121 (2011) 1013–1043
stochastic processes and their applications
www.elsevier.com/locate/spa

Rates of convergence in the central limit theorem for linear statistics of martingale differences

Jérôme Dedecker[a,*], Florence Merlevède[b]

[a] Université Paris Descartes, Laboratoire MAP5, UMR CNRS 8145, 45 rue des Saints-Pères, F-75270 Paris cedex 06, France
[b] Université Paris Est-Marne la Vallée, LAMA and C.N.R.S. UMR 8050, 5 Boulevard Descartes, 77 454 Marne La Vallée cedex 2, France

Received 17 December 2009; received in revised form 16 December 2010; accepted 16 January 2011
Available online 22 January 2011

Abstract

In this paper, we give rates of convergence for minimal distances between linear statistics of martingale differences and the limiting Gaussian distribution. In particular the results apply to the partial sums of (possibly long range dependent) linear processes, and to the least squares estimator in some parametric regression models.

MSC: 60 F 05; 60 G 10

Keywords: Minimal distances; Ideal distances; Gaussian approximation; Martingales; Linear processes; Long range dependence

1. Introduction and notation

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be probability space, and let $T : \Omega \mapsto \Omega$ be a bijective bimeasurable transformation preserving the probability $\mathbb{P}$. For a subfield $\mathcal{F}_0$ satisfying $\mathcal{F}_0 \subseteq T^{-1}(\mathcal{F}_0)$, let $\mathcal{F}_i = T^{-i}(\mathcal{F}_0)$, $\mathcal{F}_{-\infty} = \bigcap_{n\geq 0} \mathcal{F}_{-n}$ and $\mathcal{F}_\infty = \bigvee_{k\in\mathbb{Z}} \mathcal{F}_k$. Let ξ_0 be a square integrable random variable such that: ξ_0 is $\mathcal{F}_0$-measurable, $\mathbb{E}(\xi_0|\mathcal{F}_{-1}) = 0$, and $\sigma^2 = \mathbb{E}(\xi_0)^2 > 0$. Define then

* Corresponding author.
E-mail addresses: jerome.dedecker@parisdescartes.fr (J. Dedecker), florence.merlevede@univ-mlv.fr (F. Merlevède).

0304-4149/$ - see front matter
doi:10.1016/j.spa.2011.01.005

$\xi_i = \xi_0 \circ T^i$, in such a way that $(\xi_i)_{i\in\mathbb{Z}}$ is a strictly stationary sequence of square integrable martingale differences adapted to the filtration $(\mathcal{F}_i)_{i\in\mathbb{Z}}$.

Let $(c_{n,i})_{i\in\mathbb{Z},n\geq 1}$ be a double-indexed sequence of real numbers such that for all $n \geq 1$, the sequence $(c_{n,i})_{i\in\mathbb{Z}}$ is in ℓ^2 (i.e. $\sum_{i\in\mathbb{Z}} c_{n,i}^2 < \infty$). For any $n \geq 1$ we consider the following linear statistic:

$$S_n = \sum_{i\in\mathbb{Z}} c_{n,i}\xi_i. \tag{1.1}$$

Many random evolutions and statistical procedures, such as parametric or nonparametric estimation of non-linear regression with fixed design, produce linear statistics of type (1.1) (see for instance Chapter 9 in Beran [1] for the case of parametric regression, or the paper by Robinson [14] where kernel estimators are used for nonparametric regression). For instance, consider the model

$$Y_k = x_k'\beta + \xi_k, \quad k = 1, \ldots, n \tag{1.2}$$

where Y_k is observed, $x_k' = (x_{k1}, \ldots, x_{kp})$ is a $1 \times p$ deterministic vector, $\beta := (\beta_1, \beta_2, \ldots, \beta_p)'$ is the parameter of interest, and $(\xi_k)_{k\in\mathbb{Z}}$ is the unobservable error process. Let $\hat{\beta}$ the least squares estimator of β. If we are interested by the asymptotic behavior of $\hat{\beta} - \beta$, then we are led to study statistics of the type (1.1), for which $c_{n,i} = 0$ if $i \notin \{1, \ldots, n\}$.

Let now $(X_k)_{k\in\mathbb{Z}}$ be a strictly stationary sequence of square integrable random variables, and assume that it is *regular* in the sense that it may be written as

$$X_k = \sum_{j\geq 0} a_j\xi_{k-j}, \tag{1.3}$$

where $(\xi_i)_{i\in\mathbb{Z}}$ is a strictly stationary sequence of orthogonal random variables (the innovation process), and $(a_i)_{i\geq 0}$ is in ℓ^2. Again the partial sum $\sum_{i=1}^n X_i$ is a linear statistic of the type (1.1), with $c_{n,i} = \sum_{k=1\vee i}^n a_{k-i}$ if $i \leq n$ and $c_{n,i} = 0$ elsewhere. In this context, assuming that the innovation process $(\xi_i)_{i\in\mathbb{Z}}$ is a sequence of independent and identically distributed (iid) random variables is often too restrictive. For many time series, the assumption that $\mathbb{E}(\xi_i|\mathcal{F}_{i-1}) = 0$ is much more realistic (think for instance of ARCH innovations): it means exactly that the best linear predictor is in fact the best predictor in the least squares sense (see [9]).

Concerning the asymptotic behavior of S_n defined by (1.1), Hannan [8] showed that if

$$\mathbb{E}(\xi_0^2|\mathcal{F}_{-\infty}) = \sigma^2 \text{ almost surely,} \tag{1.4}$$

and

$$B_n := \frac{\max_{j\in\mathbb{Z}} |c_{n,j}|}{v_n} \text{ tends to 0 as } n \text{ tends to infinity,} \tag{1.5}$$

where

$$v_n^2 = \sum_{j\in\mathbb{Z}} c_{n,j}^2, \tag{1.6}$$

then $v_n^{-1}S_n$ converges in distribution to the normal law $\mathcal{N}(0, \sigma^2)$. It is worth noting that the condition (1.4) cannot even be replaced by ergodicity (see the example 2.1 in [12]).

Denoting by P_{S_n/v_n} the law of S_n/v_n and by G_{σ^2} the normal distribution $\mathcal{N}(0, \sigma^2)$, we are interested in this paper in giving quantitative estimates for the convergence of P_{S_n/v_n} to G_{σ^2}.

We shall consider minimal distances of type W_1, which is also called the Kantorovich distance. If we denote by F_{S_n/v_n} and Φ_{σ^2} the respective distribution functions of P_{S_n/v_n} and G_{σ^2}, then

$$W_1(P_{S_n/v_n}, G_{\sigma^2}) = \int |F_{S_n/v_n}(t) - \Phi_{\sigma^2}(t)|\mathrm{d}t = \int_0^1 |F^{-1}_{S_n/v_n}(u) - \Phi^{-1}_{\sigma^2}(u)|\mathrm{d}u.$$

As a consequence of our main result, we shall see (cf. Comment (3.2)) that the rate of convergence of $W_1(P_{S_n/v_n}, G_{\sigma^2})$ to zero can be controlled by the rate of convergence of B_n to zero. For instance if

$$\xi_0 \in \mathbb{L}^3, \quad \text{and} \quad \sum_{n\geq 1} \frac{1}{n^{1/2}} \|\mathbb{E}(\xi_n^2|\mathcal{F}_0) - \sigma^2\|_{3/2} < \infty, \tag{1.7}$$

then

$$W_1(P_{S_n/v_n}, G_{\sigma^2}) = O(B_n \log(1 + B_n^{-2})).$$

As a corollary (see Comment 3.3), we obtain the following upper bound in the Berry–Esseen theorem: if (1.7) holds then

$$\|F_{S_n/v_n} - \Phi_{\sigma^2}\|_\infty = O(B_n^{1/2}\sqrt{\log(1 + B_n^{-2})}).$$

As we shall see, in many cases $B_n = O(n^{-1/2})$ leading to the fact that under (1.7),

$$\|F_{S_n/v_n} - \Phi_{\sigma^2}\|_\infty = O(n^{-1/4}\sqrt{\log(n)}). \tag{1.8}$$

In the case where $c_{n,i} = 1$ if $i \in \{1, \ldots, n\}$ and $c_{n,i} = 0$ elsewhere, that is $S_n = \sum_{i=1}^n \xi_i$, the inequality (1.8) provides the same rate of convergence (up to the $\sqrt{\log(n)}$ term) as the best known rate obtained by Jan [11] under a condition stronger than (1.7). See also Bolthausen [3] who gave a counter-example (for non-stationary ξ_i's), showing that the rate $n^{-1/4}$ in the Berry–Esseen theorem cannot be improved when S_n is a martingale. Note also that in this particular case, the condition (1.7) can be slightly weakened (see Theorem 2.1 in [4]).

2. Definition of the distances and known results

2.1. Definition of the distances

We first define the distances that we consider in this paper. Let $\mathcal{L}(\mu, \nu)$ be the set of probability laws on $\mathbb{R}^2$ with marginals μ and ν. Let us consider the following minimal distances (sometimes called Wasserstein distances of order r):

$$W_r(\mu, \nu) = \begin{cases} \inf\left\{\int |x-y|^r P(\mathrm{d}x, \mathrm{d}y) : P \in \mathcal{L}(\mu, \nu)\right\} & \text{if } 0 < r < 1 \\ \inf\left\{\left(\int |x-y|^r P(\mathrm{d}x, \mathrm{d}y)\right)^{1/r} : P \in \mathcal{L}(\mu, \nu)\right\} & \text{if } r \geq 1. \end{cases}$$

It is well known that for two probability measures μ and ν on $\mathbb{R}$ with respective distributions functions (d.f.) F and G,

$$W_r(\mu, \nu) = \left(\int_0^1 |F^{-1}(u) - G^{-1}(u)|^r \mathrm{d}u\right)^{1/r} \quad \text{for any } r \geq 1. \tag{2.1}$$

We consider also the following ideal distances of order r (Zolotarev distances of order r). For two probability measures μ and ν, and r a positive real, let

$$\zeta_r(\mu, \nu) = \sup\left\{\int f \mathrm{d}\mu - \int f \mathrm{d}\nu : f \in \Lambda_r\right\},$$

where Λ_r is defined as follows: denoting by l the natural integer such that $l < r \leq l + 1$, Λ_r is the class of real functions f which are continuously differentiable l times and such that

$$|f^{(l)}(x) - f^{(l)}(y)| \leq |x - y|^{r-l} \quad \text{for any } (x, y) \in \mathbb{R} \times \mathbb{R}. \tag{2.2}$$

For $r \in]0, 1]$, applying the Kantorovich–Rubinstein theorem (see for instance [5], Theorem 11.8.2) to the metric $d(x, y) = |x - y|^r$, we infer that

$$W_r(\mu, \nu) = \zeta_r(\mu, \nu). \tag{2.3}$$

For probability laws on the real line, Rio [13] proved that for any $r > 1$,

$$W_r(\mu, \nu) \leq c_r\big(\zeta_r(\mu, \nu)\big)^{1/r}, \tag{2.4}$$

where c_r is a constant depending only on r.

2.2. The iid case

Let $(X_i)_{1\leq i\leq n}$ be n independent and centered random variables in $\mathbb{L}^p$, for some $p \in]2, 3]$. Let μ_n be the law of $\sum_{i=1}^n X_i/\mathrm{Var}(\sum_{i=1}^n X_i)$. It follows from the non-uniform estimates of Bikelis [2] that

$$W_1(\mu_n, G_1) \leq C(p)\left(\mathrm{Var}\left(\sum_{i=1}^n X_i\right)\right)^{-p/2} \sum_{i=1}^n \mathbb{E}(|X_i|^p). \tag{2.5}$$

In addition, in the same context, Sakhanenko [15] proved that

$$W_p^p(\mu_n, G_1) \leq \tilde{C}(p)\left(\mathrm{Var}\left(\sum_{i=1}^n X_i\right)\right)^{-p/2} \sum_{i=1}^n \mathbb{E}(|X_i|^p). \tag{2.6}$$

By Holder's inequality, we have that for any $r \in [1, p]$: $W_r^r \leq W_1^{(p-r)/(p-1)}(W_p^p)^{(r-1)/(p-1)}$. Consequently, combining (2.5) and (2.6), we get that for independent and non-identically distributed random variables with moments of order $p \in]2, 3]$, and for any $r \in [1, p]$,

$$W_r^r(\mu_n, G_1) \leq C_{p,r}\left(\mathrm{Var}\left(\sum_{i=1}^n X_i\right)\right)^{-p/2} \sum_{i=1}^n \mathbb{E}(|X_i|^p). \tag{2.7}$$

This estimate in the case of linear statistics of type (1.1) leads to the following result.

Corollary 2.1. *Let $p \in]2, 3]$. Assume that $(\xi_i)_{i\in\mathbb{Z}}$ is a sequence of iid random variables in $\mathbb{L}^p$, with $\mathbb{E}(\xi_0) = 0$ and $\mathbb{E}(\xi_0^2) = \sigma^2$. Let S_n be defined by (1.1), and v_n be defined by (1.6). For any $r \in [1, p]$, there exists a positive constant C such that for every positive integer n,*

$$W_r^r(P_{S_n}, G_{v_n^2\sigma^2}) \leq C L_{p,r}(n), \tag{2.8}$$

where

$$L_{p,r}(n) := \frac{\sum_{j \in \mathbb{Z}} |c_{n,j}|^p}{v_n^{p-r}}.$$

The proof of this result will be given in Section 6.1.

3. Main results

In this section we shall give two upper bounds for the quantity $\zeta_r(P_{S_n}, G_{v_n^2\sigma^2})$ when $(\xi_i)_{i\in\mathbb{Z}}$ is a strictly stationary sequence of martingale differences in $\mathbb{L}^p$ for $p \in]2, 3]$. The results of this section are proved in Sections 6.2–6.5.

Theorem 3.1. *Let $p \in]2, 3]$. Assume that $(\xi_i)_{i\in\mathbb{Z}}$ is a strictly stationary sequence of martingale differences in $\mathbb{L}^p$, with $\mathbb{E}(\xi_0^2) = \sigma^2$. Let $r \in [p-2, p]$ and $\alpha \in [0, p-2]$, and assume that*

$$\sum_{n\geq 1} \frac{1}{n^{1-\beta}} \|\mathbb{E}(\xi_n^2|\mathcal{F}_0) - \sigma^2\|_{p/2} < \infty, \quad \text{where } \beta = \frac{\alpha}{2} + \left(\frac{2p-\alpha-r}{2}\right)\left(\frac{p-\alpha-2}{p-\alpha}\right). \tag{3.1}$$

Let S_n be defined by (1.1), and v_n be defined by (1.6). There exists a positive constant C such that for every positive integer n,

$$\zeta_r(P_{S_n}, G_{v_n^2\sigma^2}) \leq C\Big(\max_{j\in\mathbb{Z}} |c_{n,j}|^r + \widetilde{L}_{p,r,\alpha}(n)\Big), \tag{3.2}$$

where

$$\widetilde{L}_{p,r,\alpha}(n) := \max_{j\in\mathbb{Z}} |c_{n,j}|^\alpha \sum_{k\in\mathbb{Z}} \frac{|c_{n,k}|^{p-\alpha}}{\left(\max_{j\in\mathbb{Z}} |c_{n,j}|^2 + \sum_{j=k+1}^{\infty} c_{n,j}^2\right)^{(p-r)/2}}. \tag{3.3}$$

Comment 3.1. In the case where $r = p$, choosing $\alpha = 0$ in (3.1), we derive that

$$\zeta_p(P_{S_n}, G_{v_n^2\sigma^2}) \leq C \sum_{j\in\mathbb{Z}} |c_{n,j}|^p, \tag{3.4}$$

as soon as

$$\sum_{n\geq 1} \frac{1}{n^{2-p/2}} \|\mathbb{E}(\xi_n^2|\mathcal{F}_0) - \sigma^2\|_{p/2} < \infty. \tag{3.5}$$

Using (2.4), we see that if $r = p$, we obtain the same upper bound as in (2.8).

Note that the quantity $\widetilde{L}_{p,r,\alpha}(n)$ can be bounded in all cases as follows:

Lemma 3.1. *Let $p \in]2, 3]$, $r \in [p-2, p]$ and $\alpha \in [0, p-2]$. Then there exists a positive constant C such that for every positive integer n,*

$$\widetilde{L}_{p,r,\alpha}(n) \leq C L^*_{p,r,\alpha}(n), \tag{3.6}$$

where

$$L^*_{p,r,\alpha}(n) = \begin{cases} \max_{j\in\mathbb{Z}} |c_{n,j}|^r \left((\max_{j\in\mathbb{Z}} |c_{n,j}|)^{\alpha-p} \sum_{j\in\mathbb{Z}} |c_{n,j}|^{p-\alpha} \right)^{(2-p+r)/2} \\ \quad \text{if } r \in]p-2, p] \\ \max_{j\in\mathbb{Z}} |c_{n,j}|^{p-2} \log\left(1 + 2(\max_{j\in\mathbb{Z}} |c_{n,j}|)^{\alpha-p} \sum_{j\in\mathbb{Z}} |c_{n,j}|^{p-\alpha} \right) \quad \text{if } r = p-2. \end{cases} \tag{3.7}$$

Comment 3.2. Using the above lemma, and choosing $\alpha = p-2$ in (3.1), we then deduce that

$$\zeta_r(P_{S_n}, G_{v_n^2\sigma^2}) \le \begin{cases} C \max_{j\in\mathbb{Z}} |c_{n,j}|^{p-2} v_n^{r-p+2} & \text{if } r \in]p-2, p] \\ C \max_{j\in\mathbb{Z}} |c_{n,j}|^{p-2} \log\Big(1 + 2(\max_{j\in\mathbb{Z}} |c_{n,j}|)^{-2} v_n^2\Big) & \text{if } r = p-2, \end{cases} \tag{3.8}$$

as soon as (3.5) holds.

As we shall see in Section 5, the quantity $\max_{j\in\mathbb{Z}} |c_{n,j}|$ in the bound (3.2) can be too big compared to $L_{p,r}(n)$ or to $\widetilde{L}_{p,r,\alpha}(n)$. In the following theorem, we replace $\max_{j\in\mathbb{Z}} |c_{n,j}|$ by another quantity allowing us to attain better rates of convergence. As a counterpart, the condition (3.9) that we impose on the sequence $(\xi_i)_{i\in\mathbb{Z}}$ is different than (3.1). Notice however than even if the conditions (3.1) and (3.9) cannot be compared, the condition (3.1) is usually more flexible in most of the applications.

Theorem 3.2. *Let $p \in]2, 3]$. Assume that $(\xi_i)_{i\in\mathbb{Z}}$ is a strictly stationary sequence of martingale differences in $\mathbb{L}^p$, with $\mathbb{E}(\xi_0^2) = \sigma^2$. Assume that*

$$\sum_{n\ge 1} \|(|\xi_0|^{p-2} \vee 1)|\mathbb{E}(\xi_n^2|\mathcal{F}_0) - \sigma^2|\|_1 < \infty. \tag{3.9}$$

Let S_n be defined by (1.1), and v_n be defined by (1.6). Then for any $r \in [p-2, p]$ and any sequence $(M_n)_{n\in\mathbb{Z}}$ of positive real numbers, there exists a positive constant C such that for every positive integer n,

$$\zeta_r(P_{S_n}, G_{v_n^2\sigma^2}) \le C(M_n^r + \widehat{L}_{p,r}(n)), \tag{3.10}$$

where

$$\widehat{L}_{p,r}(n) := \sum_{k\in\mathbb{Z}} \frac{|c_{n,k}|^p}{\left(M_n^2 + \sum_{j=k+1}^{\infty} c_{n,j}^2\right)^{(p-r)/2}}. \tag{3.11}$$

Comment 3.3. According to the equality (2.3) and to Remark 2.4 of Dedecker et al. [4], we get that for any $p \in]2, 3]$,

$$\|F_{S_n/v_n} - \Phi_{\sigma^2}\|_\infty \le (1 + \sigma^{-1}(2\pi)^{-1/2})\big(\zeta_{p-2}(P_{S_n/v_n}, G_{\sigma^2})\big)^{1/(p-1)},$$

where F_{S_n/v_n} is the distribution function of $v_n^{-1} S_n$ and Φ_{σ^2} is the distribution function of G_{σ^2}. Consequently Theorems 3.1 and 3.2 also give rates of convergence in terms of the uniform distance.

Comment 3.4. In the case where $r = p - 2$ and $(r, p) \neq (1, 3)$, we shall prove in Section 6.5 that the following bound is also valid: if (3.9) holds, then

$$\zeta_r(P_{S_n}, G_{v_n^2\sigma^2}) \leq C \max_{j\in\mathbb{Z}} |c_{n,j}|^{p-2}. \tag{3.12}$$

4. Application to linear processes

4.1. Linear processes with martingale difference innovations

We consider here the linear process

$$X_k = \sum_{i\in\mathbb{Z}} a_i \xi_{k-i} \quad \text{where } (a_i)_{i\in\mathbb{Z}} \in \ell^2, \tag{4.1}$$

and $(\xi_i)_{i\in\mathbb{Z}}$ is a strictly stationary sequence of martingale differences such that $\sigma^2 = \mathbb{E}(\xi_0)^2 > 0$.

As already mentioned in the introduction the partial sum $S_n = \sum_{i=1}^n X_i$ is of the form (1.1) with

$$c_{n,i} = a_{1-i} + \cdots + a_{n-i}. \tag{4.2}$$

In general, the covariances of $(X_k)_{k\in\mathbb{Z}}$ may not be summable, so the linear process may exhibit long range dependence, and therefore the variance of S_n may not be linear in n. In fact, the variance of S_n is equal to $\sigma^2 v_n^2$, where v_n is defined by (1.6):

$$v_n^2 = \sum_{j\in\mathbb{Z}} c_{n,j}^2 = \sum_{j\in\mathbb{Z}} \left(\sum_{k=1}^n a_{k-j} \right)^2. \tag{4.3}$$

The following result follows straightforwardly from Theorem 3.1 and Comment 3.2.

Corollary 4.1. *Let $p \in]2, 3]$. Let $(X_k)_{k\in\mathbb{Z}}$ be defined by (4.1), and assume that $(\xi_i)_{i\in\mathbb{Z}}$ satisfies (3.5). Let $c_{n,i}$ be defined by (4.2) and v_n be defined by (4.3). Assume also that there exists a positive constant K such that for every positive integer n,*

$$\max_{j\in\mathbb{Z}} |c_{n,j}| \leq K \frac{v_n}{\sqrt{n}}. \tag{4.4}$$

Then there exists a positive constant C such that for every positive integer n,

$$\zeta_r(P_{S_n/v_n}, G_{\sigma^2}) \leq \begin{cases} Cn^{1-p/2} & \text{if } r \in]p-2, p] \\ Cn^{1-p/2} \log n & \text{if } r = p-2. \end{cases} \tag{4.5}$$

This result deserves some comments:

Comment 4.1. In the case where $r \in]p - 2, p]$, under the assumptions of the previous corollary, we get the same rate of convergence as for a sum of n iid random variables in $\mathbb{L}^p$.

Comment 4.2. Condition (4.4) holds in a lot of situations. Let us briefly describe two of them:

1. Weak dependence: if the spectral density f of $(X_k)_{k\in\mathbb{Z}}$ is continuous at 0, then $n^{-1}\mathrm{Var}(S_n)$ converges to $2\pi f(0)$ (see [7], Corollary 4, page 228). If moreover $f(0) > 0$, we infer that

$v_n/\sqrt{n}$ converges to $\sigma^{-1}\sqrt{2\pi f(0)} > 0$. In that case Condition (4.4) holds if and only if

$$\sup_{n>0}\sup_{i\in\mathbb{Z}}\left|\sum_{k=i+1}^{i+n} a_i\right| < \infty.$$

In particular, Condition (4.4) holds as soon as $\sum_{i\in\mathbb{Z}}|a_i| < \infty$ and $\sum_{i\in\mathbb{Z}} a_i \neq 0$.

2. Long range dependence: for the selection $a_i = 0$ for $i < 0$ and $a_i = i^{-\alpha}\ell(i)$ for $i > 0$, where ℓ is a slowly varying function at infinity and $1/2 < \alpha < 1$, then $v_n^2 \sim \kappa_\alpha n^{3-2\alpha}\ell^2(n)$ where κ_α is a positive constant depending on α. Using the properties of slowly varying functions, it is easy to see that Condition (4.4) is verified. To give an example of a linear process satisfying such assumptions, we mention the fractionally integrated processes. These models play an important role in financial econometrics, climatology and so on, and are widely studied. For $0 < d < 1/2$, let

$$X_k = (1-B)^{-d}\xi_k = \sum_{i\geq 0} a_i\xi_{k-i} \text{ where } a_i = \frac{\Gamma(i+d)}{\Gamma(d)\Gamma(i+1)}, \tag{4.6}$$

where B is the lag operator. If $0 < d < 1/2$, the covariances of $(X_k)_{k\in\mathbb{Z}}$ are not summable, the variance of partial sums is asymptotically proportional to n^{2d+1} and the linear process exhibits long range dependence. In addition since $a_i \sim \kappa_\alpha i^{-\alpha}$ with $\alpha = 1-d$, these processes satisfy (4.4) for any $0 < d < 1/2$.

Comment 4.3. Let F_{S_n/v_n} be the distribution function of $v_n^{-1}S_n$ and Φ_{σ^2} be the distribution function of G_{σ^2}. According to Comment 3.3, we get the following bound in the Berry–Esseen theorem: under the conditions of Corollary 4.1,

$$\|F_{S_n/v_n} - \Phi_{\sigma^2}\|_\infty \leq Cn^{-\frac{p-2}{2(p-1)}}(\log n)^{1/(p-1)}.$$

Comment 4.4. If we do not impose Condition (4.4), then under the assumptions of Corollary 4.1 on the sequence $(\xi_i)_{i\in\mathbb{Z}}$, we derive the following rates of convergence:

$$\zeta_r(P_{S_n/v_n}, G_{\sigma^2}) \leq \begin{cases} CB_n^{p-2} & \text{if } r \in]p-2, p] \\ CB_n^{p-2}\log(1+B_n^{-2}) & \text{if } r = p-2, \end{cases} \tag{4.7}$$

where

$$B_n = \frac{\max_{j\in\mathbb{Z}}|c_{n,j}|}{v_n}. \tag{4.8}$$

This still gives rates of convergence as soon as v_n tends to infinity, since the following universal bound is valid for B_n: there exists a positive constant K such that

$$B_n \leq K(1+v_n^{1/2})v_n^{-1}. \tag{4.9}$$

The upper bound (4.9) has been proved by Robinson [14], Lemma 2(ii).

Comment 4.5. Corollary 4.1 applies to the case where $(\xi_i)_{i\in\mathbb{Z}}$ has an ARCH(∞) structure as described by Giraitis et al. [6], that is

$$\xi_n = \sigma_n\eta_n, \quad \text{with } \sigma_n^2 = c + \sum_{j=1}^{\infty} c_j\xi_{n-j}^2, \tag{4.10}$$

where $(\eta_n)_{n\in\mathbb{Z}}$ is a sequence of iid centered random variables such that $\mathbb{E}(\eta_0^2) = 1$, and where $c \geq 0$, $c_j \geq 0$, and $\sum_{j\geq 1} c_j < 1$. In that case, we shall prove in Section 6.6 that the condition (3.5) holds as soon as $\|\eta_0\|_p < \infty$ and

$$\|\eta_0\|_p^2 \sum_{j\geq 1} c_j < 1 \quad \text{and} \quad \sum_{j\geq n} c_j = O(n^{-b}) \quad \text{for } b > p/2 - 1. \tag{4.11}$$

4.2. Linear processes with dependent innovations

In this section, we no longer assume that $\mathbb{E}(\xi_i|\mathcal{F}_{i-1}) = 0$, but now that ξ_i can be approximated by a martingale difference d_j satisfying the assumptions of Corollary 4.1. The following result is proved in Section 6.7.

Theorem 4.1. *Let $p \in]2, 3]$. Let $(\xi_i)_{i\in\mathbb{Z}} = (\xi_0 \circ T^i)_{i\in\mathbb{Z}}$ be a stationary sequence of centered random variables in $\mathbb{L}^p$ such that*

$$\sum_{k>0} \mathbb{E}(\xi_k|\mathcal{F}_0) \quad \textit{and} \quad \sum_{k>0} \xi_{-k} - \mathbb{E}(\xi_{-k}|\mathcal{F}_0) \quad \textit{converge in } \mathbb{L}^p. \tag{4.12}$$

Let $(X_k)_{k\in\mathbb{Z}}$ be defined by (4.1), $S_n = \sum_{k=1}^n X_k$, and let v_n be defined by (4.3). For any $j \in \mathbb{Z}$, let

$$d_j = \sum_{k\in\mathbb{Z}} \mathbb{E}(\xi_k|\mathcal{F}_j) - \mathbb{E}(\xi_k|\mathcal{F}_{j-1}).$$

Let $\sigma^2 = \mathbb{E}(d_0^2) = \sum_{k\in\mathbb{Z}} \mathbb{E}(\xi_0\xi_k)$ and $\sigma_n^2 = v_n^{-2}\mathbb{E}(S_n)^2$. If $(d_j)_{j\in\mathbb{Z}}$ satisfies (3.5), and if there exists a positive constant C such that for every positive integer n,

$$|c_{n,0}| + \sum_{j\in\mathbb{Z}} |a_{n+j} - a_j| \leq C\frac{v_n}{\sqrt{n}}, \tag{4.13}$$

then

1. $\zeta_{p-2}(P_{S_n/v_n}, G_{\sigma^2}) = O(n^{1-p/2}\log n)$,
2. $\zeta_r(P_{S_n/v_n}, G_{\sigma^2}) = O(n^{1-p/2})$ *if* $r \in]p-2, 2]$,
3. $\zeta_r(P_{S_n/v_n}, G_{\sigma_n^2}) = O(n^{1-p/2})$ *if* $r \in]2, p]$.

Comment 4.6. The results of items 1 and 2 are valid with σ_n instead of σ. In contrast, the result of item 3 is no longer true if σ_n is replaced by σ, because for $r \in]2, 3]$, a necessary condition for $\zeta_r(\mu, \nu)$ to be finite is that the first two moments of ν and μ are equal. Note that, under the assumptions of Theorem 4.1, both $W_r(P_{S_n/v_n}, G_{\sigma_n^2})$ and $W_r(P_{S_n/v_n}, G_{\sigma^2})$ are $O(n^{-(p-2)/2\max(1,r)})$ for $r \in]p-2, p]$. Indeed, for $r \in]2, p]$, it suffices to note that

$$W_r(P_{S_n/v_n}, G_{\sigma^2}) \leq W_r(P_{S_n/v_n}, G_{\sigma_n^2}) + W_r(G_{\sigma_n^2}, G_{\sigma^2}),$$

and the second term on the right is of order $|\sigma - \sigma_n| = O(n^{-1/2})$ (to see this, use (4.13) and the inequality (6.44) in Section 6.7).

Comment 4.7. Condition (4.13) implies Condition (4.4). As for (4.4), it holds in a lot of situations. Let us briefly describe two of them:

1. Weak dependence: if $\sum_{i\in\mathbb{Z}} |a_i| < \infty$ and $\sum_{i\in\mathbb{Z}} a_i \neq 0$ then (4.13) holds.

2. Long range dependence: if $a_i = 0$ for $i < 0$, and $a_i \sim i^{-\alpha}$ as $i \to \infty$, with $1/2 < \alpha < 1$, then (4.13) holds.

Comment 4.8. Let us give some examples of stationary sequences $(\xi_i)_{i\in\mathbb{Z}}$ for which (4.12) holds, and $(d_j)_{j\in\mathbb{Z}}$ satisfies (3.5). We follow here the approach of Wu [17], Section 3.

Let $(\epsilon_i)_{i\in\mathbb{Z}}$ be a sequence of iid random variables, and let $\mathcal{F}_i = \sigma(\epsilon_k, k \le i)$. Let $(\epsilon'_i)_{i\in\mathbb{Z}}$ be an independent copy of $(\epsilon_i)_{i\in\mathbb{Z}}$. Let $Y_n = (\ldots, \epsilon_{n-1}, \epsilon_n)$, and for $n \ge 0$, $Y_n^* = (\ldots, \epsilon'_{-1}, \epsilon'_0, \epsilon_1, \ldots, \epsilon_n)$. Assume that the random variables $\xi_n = g(Y_n)$ are well defined, centered, and in $\mathbb{L}^p$, and let

$$\beta^*(n) = \|g(Y_n) - g(Y_n^*)\|_p.$$

From Proposition 3 in Wu [17], we infer that (4.12) holds, and $(d_j)_{j\in\mathbb{Z}}$ satisfies (3.5) as soon as

$$\sum_{k=1}^{\infty} k^{(p-2)/2}\beta^*(k) < \infty. \tag{4.14}$$

Comment 4.9. Comment 4.8 applies to the causal linear process $\xi_n = \sum_{i\ge 0} b_i\epsilon_{n-i}$, but, as we shall see, the condition (4.14) is suboptimal in that case. Let us consider the general case: $(\epsilon_i)_{i\in\mathbb{Z}}$ is a sequence of iid random variables with $\mathbb{E}(\epsilon_0) = 0$ and $\|\epsilon_0\|_p < \infty$, and

$$\xi_n = \sum_{i\in\mathbb{Z}} b_i\epsilon_{n-i}, \quad \text{where } (b_i)_{i\in\mathbb{Z}} \in \ell^2.$$

Assume that the two series $\sum_{i\ge 0} b_i$ and $\sum_{i<0} b_i$ converge, and that [10] condition holds, that is

$$\sum_{n=1}^{\infty}\Bigl(\sum_{k\ge n} b_k\Bigr)^2 < \infty \quad \text{and} \quad \sum_{n=1}^{\infty}\Bigl(\sum_{k\le -n} b_k\Bigr)^2 < \infty. \tag{4.15}$$

Notice that $\mathbb{E}(\xi_k|\mathcal{F}_0) = \sum_{\ell\ge 0} b_{k+\ell}\epsilon_{-\ell}$ and that $\xi_{-k} - \mathbb{E}(\xi_{-k}|\mathcal{F}_0) = \sum_{\ell\ge 1} b_{-k-\ell}\epsilon_\ell$. From Burkholder's inequality, there exists a constant C such that for any positive integers m and n with $m < n$,

$$\begin{aligned}\Bigl\|\sum_{k=m+1}^{n}\mathbb{E}(\xi_k|\mathcal{F}_0)\Bigr\|_p^2 &\le C\|\epsilon_0\|_p^2\sum_{\ell=0}^{\infty}\Bigl(\sum_{k=m+1}^{n} b_{k+\ell}\Bigr)^2\\ &\le 2C\|\epsilon_0\|_p^2\left(\sum_{\ell=m+1}^{\infty}\Bigl(\sum_{k\ge\ell} b_k\Bigr)^2 + \sum_{\ell=n+1}^{\infty}\Bigl(\sum_{k\ge\ell} b_k\Bigr)^2\right),\end{aligned}$$

which converges to zero under the first part of (4.15) as m and n tend to infinity. Similarly we derive that there exists a constant C such that for any positive integers m and n with $m < n$,

$$\Bigl\|\sum_{k=m+1}^{n}\bigl(\xi_{-k} - \mathbb{E}(\xi_{-k}|\mathcal{F}_0)\bigr)\Bigr\|_p^2 \le 2C\|\epsilon_0\|_p^2\left(\sum_{\ell=m+2}^{\infty}\Bigl(\sum_{k\le-\ell} b_k\Bigr)^2 + \sum_{\ell=n+2}^{\infty}\Bigl(\sum_{k\le-\ell} b_k\Bigr)^2\right),$$

which converges to zero under the second part of (4.15) as m and n tend to infinity. From these considerations, we derive that (4.12) holds. Now $d_j = \epsilon_j\sum_{\ell\in\mathbb{Z}} b_\ell$ and the ϵ_i's are iid, so (3.5) is satisfied.

Notice that (4.15) holds if either $\sum_{i\in\mathbb{Z}} i^2 b_i^2 < \infty$ or $\sum_{i\in\mathbb{Z}} \sqrt{|i|}|b_i| < \infty$. By contrast, if $b_i = 0$ for $i < 0$, the condition (4.14) is true as soon as

$$\sum_{n=1}^{\infty} n^{(p-2)/2} \left(\sum_{k\geq n} b_k^2 \right)^{1/2} < \infty,$$

which is always stronger than (4.15), since it implies that $\sum_{i>0} i^2 b_i^2 < \infty$.

5. Application to parametric regression

Let us consider the simple parametric regression model

$$Y_i = \beta \alpha_i + \xi_i,$$

where $(\xi_i)_{i\in\mathbb{Z}}$ is a stationary sequence of martingale differences such that $\mathbb{E}(\xi_0^2) = \sigma^2$, $(\alpha_i)_{i\geq 1}$ is a sequence of real numbers such that $\sum_{i=1}^n \alpha_i^2$ tends to infinity, and β is the parameter of interest. The least squares estimator $\hat{\beta}$ of β satisfies

$$S_n = \hat{\beta} - \beta = \frac{\sum_{i=1}^n \alpha_i \xi_i}{\sum_{j=1}^n \alpha_j^2}.$$

Consequently, if $(\max_{i\in[1,n]} |\alpha_i|)(\sum_{j=1}^n \alpha_j^2)^{-1}$ tends to 0, Theorem 3.1 (resp. Theorem 3.2) applied with

$$c_{n,i} = \frac{\alpha_i}{\sum_{j=1}^n \alpha_j^2} \mathbf{1}_{1\leq i\leq n} \tag{5.1}$$

gives rates of convergence for the normal approximation of $S_n = \hat{\beta} - \beta$ in terms of minimal distances as soon as $(\xi_i)_{i\in\mathbb{Z}}$ satisfies (3.1) (resp. (3.9)).

For instance, the following corollary holds:

Corollary 5.1. *Let $p \in]2, 3]$. Let $c_{n,j}$ be defined by (5.1) and v_n be defined by (1.6). Assume that $|\alpha_n|$ is non-decreasing, and satisfies*

$$\limsup_{n\to\infty} \frac{|\alpha_n|}{|\alpha_{[n/2]}|} \leq C.$$

If $(\xi_i)_{i\in\mathbb{Z}}$ satisfies (3.5), then

$$\zeta_r(P_{S_n/v_n}, G_{\sigma^2}) \leq \begin{cases} Cn^{1-p/2} & \text{if } r \in]p-2, p] \\ Cn^{1-p/2}\log(n) & \text{if } r = p-2. \end{cases}$$

Comment 5.1. Note that, if α_n satisfies the conditions of the above corollary, then the Lyapunov coefficient $v_n^{-r} L_{p,r}(n)$ defined in (2.8) is such that

$$C_1 n^{1-p/2} \leq v_n^{-r} L_{p,r}(n) \leq C_2 n^{1-p/2}.$$

It follows from Corollary 2.1 and (2.4) that for $r \in [1, p]$ and $(r, p) \neq (1, 3)$ we obtain the same rate of convergence for $W_r(P_{S_n/v_n}, G_{\sigma^2})$ as in the case where $(\xi_i)_{i\in\mathbb{Z}}$ is iid.

Now if $|\alpha_n|$ decreases to zero, the quantity $v_n^{-r} L_{p,r}(n)$ given in Corollary 2.1 depends on the rate of convergence of α_n to zero. For instance, if $\alpha_i = i^{-\gamma}$ with $0 < \gamma < 1/2$, we have

$$v_n^{-r} L_{p,r}(n) \leq \begin{cases} Cn^{(2\gamma-1)p/2} & \text{if } \gamma p > 1 \\ Cn^{1-p/2}\log(n) & \text{if } \gamma p = 1 \\ Cn^{1-p/2} & \text{if } \gamma p < 1. \end{cases} \tag{5.2}$$

In the case $\gamma p > 1$, the rate given above can never be attained by applying Theorem 3.1, except if $r = p$. This is mainly due to the fact that the rate given by Theorem 3.1 cannot be better than $v_n^r \sim Cn^{(2\gamma-1)r/2}$.

In Section 6.8, we shall prove the following corollary. It shows that, choosing $M_n = \alpha_n(\sum_{i=1}^n \alpha_i^2)^{-1}$ in Theorem 3.2, one recovers the rates given in (5.2) in the case where $r > p-2$ and also in the case where $r = p - 2$ and $\gamma p \geq 1$.

Corollary 5.2. *Let $p \in]2, 3]$. Let $c_{n,j}$ be defined by (5.1) and v_n be defined by (1.6). Let $\alpha_i = i^{-\gamma}$ for $0 < \gamma < 1/2$ and assume that $(\xi_i)_{i\in\mathbb{Z}}$ satisfies (3.9). If $r \in]p-2, p]$, there exists a positive constant C such that for every positive integer n,*

$$\zeta_r(P_{S_n/v_n}, G_{\sigma^2}) \leq \begin{cases} Cn^{(2\gamma-1)p/2} & \text{if } \gamma p > 1 \\ Cn^{1-p/2}\log(n) & \text{if } \gamma p = 1 \\ Cn^{1-p/2} & \text{if } \gamma p < 1. \end{cases} \tag{5.3}$$

For $r = p - 2$, there exists a positive constant C such that for every positive integer n,

$$\zeta_{p-2}(P_{S_n/v_n}, G_{\sigma^2}) \leq \begin{cases} Cn^{(2\gamma-1)p/2} & \text{if } \gamma p > 1 \\ Cn^{1-p/2}\log(n) & \text{if } \gamma p \leq 1. \end{cases} \tag{5.4}$$

6. Proofs

From now on, we denote by C a numerical constant which may vary from line to line. Let us introduce the following notation:

Notation 6.1. For l integer, q in $]l, l+1]$ and f continuously differentiable l times, we set

$$|f|_{\Lambda_q} = \sup\{|x-y|^{l-q}|f^{(l)}(x) - f^{(l)}(y)| : (x, y) \in \mathbb{R}\times\mathbb{R}\}.$$

6.1. *Proof of Corollary 2.1*

For two positive integer L and K, we set $S_{n,K,L} = \sum_{j=-K}^{L} c_{n,j}\xi_j$ and $v_{n,K,L}^2 = \sum_{j=-K}^{L} c_{n,j}^2$. We have that

$$W_r(P_{S_n}, G_{v_n^2\sigma^2}) \leq \liminf_{K,L\to\infty}\big(W_r(P_{S_n}, P_{S_{n,K,L}}) + W_r(P_{S_{n,K,L}}, G_{v_{n,K,L}^2\sigma^2}) + W_r(G_{v_{n,K,L}^2\sigma^2}, G_{v_n^2\sigma^2})\big).$$

Using (2.7), we get that $W_r^r(P_{S_{n,K,L}}, G_{v_{n,K,L}^2\sigma^2}) \leq C_{p,r} v_{n,K,L}^{r-p} \sum_{j=-K}^{L} |c_{n,j}|^p$. Hence

$$\liminf_{K,L\to\infty} W_r^r(P_{S_{n,K,L}} G_{v_{n,K,L}^2\sigma^2}) \leq C_{p,r} L_{p,r}(n).$$

Hence the result will follow if we can prove that

$$\lim_{K,L\to\infty} W_r(P_{S_n}, P_{S_{n,K,L}}) = 0, \quad \text{and} \quad \lim_{K,L\to\infty} W_r(G_{v_{n,K,L}^2\sigma^2}, G_{v_n^2\sigma^2}) = 0. \tag{6.1}$$

Since for $r \in [1, p]$,

$$W_r(G_{v_{n,K,L}^2\sigma^2}, G_{v_n^2\sigma^2}) \le C\sigma|v_{n,K,L} - v_n| \le C\sigma\left(\sum_{j>L} c_{n,j}^2 + \sum_{j<-K} c_{n,j}^2\right)^{1/2},$$

the second part of (6.1) holds. To prove the first part, we write that

$$W_r(P_{S_n}, P_{S_{n,K,L}}) \le \|S_n - S_{n,K,L}\|_r.$$

Hence from the Burkholder inequality

$$W_r(P_{S_n}, P_{S_{n,K,L}}) \le C\|\xi_0\|_{r\vee 2}\left(\sum_{j>L} c_{n,j}^2 + \sum_{j<-K} c_{n,j}^2\right)^{1/2},$$

proving the first part of (6.1).

6.2. Proof of Theorem 3.1

For a positive integer N let $S_{n,N} = \sum_{j=1}^N c_{n,j}\xi_j$ and let $v_{n,N}^2 = \sum_{j=1}^N c_{n,j}^2$. We first show that without restricting the generality, it suffices to prove that for any positive integer N,

$$\zeta_r(P_{S_{n,N}}, G_{v_{n,N}^2\sigma^2}) \le C\big(\max_{j\in\mathbb{Z}} |c_{n,j}|^r + K_{p,r,\alpha}(n, N)\big), \tag{6.2}$$

where

$$K_{p,r,\alpha}(n, N) = \max_{j\in\mathbb{Z}} |c_{n,j}|^\alpha \sum_{k=1}^N \frac{|c_{n,k}|^{p-\alpha}}{\left(\max_{j\in\mathbb{Z}} |c_{n,j}|^2 + \sum_{j=k+1}^N c_{n,j}^2\right)^{(p-r)/2}}.$$

With this aim, for two positive integers L and K, we set $S_{n,K,L} = \sum_{j=-K}^L c_{n,j}\xi_j$ and $v_{n,K,L}^2 = \sum_{j=-K}^L c_{n,j}^2$. By the Burkholder inequality, for any $r \in [p-2, p]$,

$$\|S_n - S_{n,K,L}\|_r \le C\|\xi_0\|_{r\vee 2}\left(\sum_{j>L} c_{n,j}^2 + \sum_{j<-K} c_{n,j}^2\right)^{1/2} \quad \text{and} \quad \|S_n\|_r \le C\|\xi_0\|_{r\vee 2} v_n.$$

Following the arguments given in the proof of Lemma 5.2 of Dedecker et al. [4], we get that

$$\lim_{K,L\to\infty} \zeta_r(P_{S_n}, P_{S_{n,K,L}}) = 0$$

and similarly

$$\lim_{K,L\to\infty} \zeta_r(G_{v_n^2\sigma^2}, G_{v_{n,K,L}^2\sigma^2}) = 0,$$

by writing that $G_{v_n^2\sigma^2} = P_{T_n}$ and $G_{v_{n,K,L}^2\sigma^2} = P_{T_{n,K,L}}$ where $T_n = \sum_{j\in\mathbb{Z}} c_{n,j}Y_j$ and $T_{n,K,L} = \sum_{j=-K}^L c_{n,j}Y_j$ with $(Y_i)_{i\in\mathbb{Z}}$ a sequence of $\mathcal{N}(0, \sigma^2)$-distributed independent random variables. It follows that for $r \in [p-2, 2]$,

$$\zeta_r(P_{S_n}, G_{v_n^2\sigma^2}) \le \liminf_{K,L\to\infty} \zeta_r(P_{S_{n,K,L}}, G_{v_{n,K,L}^2\sigma^2}). \tag{6.3}$$

Consider now the case where $r \in]2, p]$. Let

$$\alpha_{n,K,L} = \frac{\|S_n\|_2}{\|S_{n,K,L}\|_2} \quad \text{and} \quad R_{n,K,L} = S_n - S_{n,K,L} + (1-\alpha_{n,K,L})S_{n,K,L}.$$

Following the arguments of the proof of Item 3 of Lemma 5.2 of Dedecker et al. [4] and using the fact that by the Burkholder inequality $\|S_{n,K,L}\|_r \le \|\xi_0\|_r v_n$, we derive that for $f \in \Lambda_r$,

$$\mathbb{E}(f(S_n) - f(\alpha_{n,K,L}S_{n,K,L})) \le \frac{1}{r-1}\alpha_{n,K,L}^{r-1}\|R_{n,K,L}\|_r v_n^{r-1} + \frac{1}{2}\alpha_{n,K,L}^{r-2}\|R_{n,K,L}\|_r^2 v_n^{r-2} + \frac{1}{2}\|R_{n,K,L}\|_r^r.$$

Since $\lim_{K,L\to\infty}\alpha_{n,K,L} = 1$ and $\lim_{K,L\to\infty}\|S_n - S_{n,K,L}\|_r = 0$, we get that $\lim_{K,L\to\infty} \|R_{n,K,L}\|_r = 0$. Consequently, for any $r \in]2, p]$, we get that

$$\lim_{K,L\to\infty} \zeta_r(P_{S_n}, P_{\alpha_{n,K,L}S_{n,K,L}}) = 0. \tag{6.4}$$

Similarly, we derive that

$$\lim_{K,L\to\infty} \zeta_r(G_{v_n^2\sigma^2}, G_{\alpha_{n,K,L}^2 v_{n,K,L}^2\sigma^2}) = 0. \tag{6.5}$$

Now notice that

$$\zeta_r(P_{\alpha_{n,K,L}S_{n,K,L}}, G_{\alpha_{n,K,L}^2 v_{n,K,L}^2\sigma^2}) = \alpha_{n,K,L}^r \zeta_r(P_{S_{n,K,L}}, G_{v_{n,K,L}^2\sigma^2}).$$

Since $\lim_{K,L\to\infty}\alpha_{n,K,L} = 1$, it follows that (6.3) also holds for $r \in]2, p]$. Let now

$$K_{p,r,\alpha}(n,K,L) = \max_{j\in\mathbb{Z}}|c_{n,j}|^\alpha \sum_{k=-K}^{L} \frac{|c_{n,k}|^{p-\alpha}}{\left(\max_{j\in\mathbb{Z}}|c_{n,j}|^2 + \sum_{j=k+1}^{L} c_{n,j}^2\right)^{(p-r)/2}}.$$

Since $\lim_{K,L\to\infty} K_{p,r,\alpha}(n,K,L) = \widetilde{L}_{p,r,\alpha}(n)$, the theorem will be proven if we can show that for any positive integers K and L,

$$\zeta_r(P_{S_{n,K,L}}, G_{v_{n,K,L}^2\sigma^2}) \le C\big(\max_{j\in\mathbb{Z}}|c_{n,j}|^r + K_{p,r,\alpha}(n,K,L)\big). \tag{6.6}$$

Since by the strict stationarity of (ξ_i), $S_{n,K,L}$ has the same distribution as $\sum_{j=1}^N c_{n,j-K+1}\xi_j$ where $N = L+K+1$, it follows that $\zeta_r(P_{S_{n,K,L}}, G_{v_{n,K,L}^2\sigma^2})$ will satisfy (6.6) as soon as (6.2) holds for any positive integer N.

We turn now to the proof of (6.2). Without loss of generality we assume that $\sigma = 1$. The general case follows by dividing the random variables by σ.

Let $(Y_i)_{i\in\mathbb{N}}$ be a sequence of $\mathcal{N}(0,1)$-distributed independent random variables, independent of the sequence $(\xi_i)_{i\in\mathbb{Z}}$. For $m > 0$, let $T_{n,m} = \sum_{j=1}^m c_{n,j}Y_j$ and $S_{n,m} = \sum_{j=1}^m c_{n,j}\xi_j$. Set $S_{n,0} = T_{n,0} = 0$. Let also Z_n be a $\mathcal{N}(0, \max_{j\in\mathbb{Z}}|c_{n,j}|^2)$-distributed random variable independent of $(\xi_i)_{i\in\mathbb{Z}}$ and $(Y_i)_{i\in\mathbb{Z}}$. Using Lemma 5.1 of Dedecker et al. [4] together with the fact that

$\zeta_r(P_{aX}, P_{aY}) = |a|^r \zeta_r(P_X, P_Y)$, we derive that for any r in $]0, p]$,

$$\zeta_r(P_{S_{n,N}}, P_{T_{n,N}}) \leq 2\zeta_r(P_{S_{n,N}} * P_{Z_n}, P_{T_{n,N}} * P_{Z_n}) + 4\sqrt{2} \max_{j\in\mathbb{Z}} |c_{n,j}|^r. \tag{6.7}$$

Consequently it remains to upper bound

$$\zeta_r(P_{S_{n,N}} * P_{Z_n}, P_{T_{n,N}} * P_{Z_n}) = \sup_{f\in\Lambda_r} \mathbb{E}(f(S_{n,N} + Z_n) - f(T_{n,N} + Z_n)).$$

For any $m \leq N$, set

$$f_{N-m,n}(x) = \mathbb{E}(f(x + T_{n,N} - T_{n,m} + Z_n)).$$

Then, from the independence of the above sequences,

$$\mathbb{E}(f(S_{n,N} + Z_n) - f(T_{n,N} + Z_n)) = \sum_{m=1}^{N} D_m, \tag{6.8}$$

where

$$D_m = \mathbb{E}\big(f_{N-m,n}(S_{n,m-1} + c_{n,m}\xi_m) - f_{N-m,n}(S_{n,m-1} + c_{n,m}Y_m)\big).$$

For any twice-differentiable function g, the Taylor integral formula at order 2 can be written as

$$g(x + h) - g(x) = g'(x)h + \frac{1}{2}h^2 g''(x) + h^2 \int_0^1 (1 - t)(g''(x + th) - g''(x))\mathrm{d}t.$$

Hence, for any q in $]2, 3]$,

$$\begin{aligned}\left|g(x + h) - g(x) - g'(x)h - \frac{1}{2}h^2 g''(x)\right| &\leq h^2 \int_0^1 (1 - t)|th|^{q-2}|g|_{\Lambda_q}\mathrm{d}t \\ &\leq \frac{1}{q(q-1)}|h|^q |g|_{\Lambda_q}.\end{aligned} \tag{6.9}$$

Let

$$D'_m = c_{n,m}^2 \mathbb{E}(f''_{N-m,n}(S_{n,m-1})(\xi_m^2 - 1)) = c_{n,m}^2 \mathbb{E}(f''_{N-m,n}(S_{n,m-1})(\xi_m^2 - Y_m^2)). \tag{6.10}$$

From (6.9) applied twice to $g = f_{N-m,n}$, $x = S_{n,m-1}$ and $h = c_{n,m}\xi_m$ or $h = c_{n,m}Y_m$ together with the martingale property,

$$\left|D_m - \frac{1}{2}D'_m\right| \leq \frac{|c_{n,m}|^p}{p(p-1)}|f_{N-m,n}|_{\Lambda_p}\mathbb{E}(|\xi_m|^p + |Y_m|^p).$$

Now $\mathbb{E}(|Y_m|^p) \leq p - 1 \leq (p-1)\mathbb{E}|\xi_0|^p$. Hence

$$R_m := |D_m - (D'_m/2)| \leq |c_{n,m}|^p \mathbb{E}|\xi_0|^p |f_{N-m,n}|_{\Lambda_p}. \tag{6.11}$$

We notice now that $|f_{N-m,n}|_{\Lambda_p} = |f * \phi_{\delta_n}|_{\Lambda_p}$ where ϕ_t be the density of the law $\mathcal{N}(0, t^2)$ and $\delta_n^2 = \mathbb{E}(Z_n + T_{n,N} - T_{n,m})^2 = \max_{j\in\mathbb{Z}} |c_{n,j}|^2 + \sum_{j=m+1}^{N} c_{n,j}^2$. Then, from Lemma 6.1 of Dedecker et al. [4], since $p \geq r$ and f belongs to Λ_r (i.e. $|f|_{\Lambda_r} \leq 1$),

$$|f_{N-m,n}|_{\Lambda_p} = |f * \phi_{\delta_n}|_{\Lambda_p} \leq C\delta_n^{r-p} = C\left(\max_{j\in\mathbb{Z}} |c_{n,j}|^2 + \sum_{j=m+1}^{N} c_{n,j}^2\right)^{(r-p)/2}. \tag{6.12}$$

Consequently,

$$\sum_{m=1}^{N} R_m \le C K_{p,r,\alpha}(n, N). \tag{6.13}$$

We now upper bound $D' = D'_1 + D'_2 + \cdots + D'_N$. For any $m = 1, \ldots, N$,

$$\begin{aligned} f''_{N-m,n}(S_{n,m-1}) = &\sum_{\ell=1}^{[\log_2 m]} \big(f''_{N-(m-2^{\ell-1})-1,n}(S_{n,m-2^{\ell-1}}) - f''_{N-(m-2^{\ell})-1,n}(S_{n,m-2^{\ell}})\big) \\ &+ f''_{N-(m-2^{[\log_2 m]})-1,n}(S_{n,m-2^{[\log_2 m]}}). \end{aligned}$$

For any $\ell = 1, \ldots, [\log_2 m]$,

$$\begin{aligned} &\mathbb{E}\Big(\big(f''_{N-(m-2^{\ell-1})-1,n}(S_{n,m-2^{\ell-1}}) - f''_{N-(m-2^{\ell})-1,n}(S_{n,m-2^{\ell}})\big)(\xi_m^2 - 1)\Big) \\ &\quad = \mathbb{E}\Big(\big(f''_{N-(m-2^{\ell-1})-1,n}(S_{n,m-2^{\ell-1}}) - f''_{N-(m-2^{\ell})-1,n}(S_{n,m-2^{\ell}})\big)\mathbb{E}\big(\xi_m^2 - 1|\mathcal{F}_{m-2^{\ell-1}}\big)\Big) \\ &\quad = \mathbb{E}\Big(\big(f''_{N-(m-2^{\ell-1})-1,n}(S_{n,m-2^{\ell-1}}) - f''_{N-(m-2^{\ell-1})-1,n} \\ &\qquad \times (S_{n,m-2^{\ell}} + T_{n,m-2^{\ell-1}+1} - T_{n,m-2^{\ell}+1})\big)\mathbb{E}\big(\xi_m^2 - 1|\mathcal{F}_{m-2^{\ell-1}}\big)\Big). \end{aligned}$$

Now (6.12) means that for any real x and y,

$$|f''_{N-i,n}(x) - f''_{N-i,n}(y)| \le C \left(\sum_{j=i+1}^{N} c_{n,j}^2 + \max_{j\in\mathbb{Z}} |c_{n,j}|^2\right)^{(r-p)/2} |x-y|^{p-2}. \tag{6.14}$$

In addition, by the Burkholder inequality,

$$\begin{aligned} &\|(S_{n,m-2^{\ell-1}} - S_{n,m-2^{\ell}}) - (T_{n,m-2^{\ell-1}+1} - T_{n,m-2^{\ell}+1})\|_p \\ &\quad \le C(\|\xi_0\|_p + \|Y_1\|_p) \left(\sum_{j=m-2^{\ell}+1}^{m-2^{\ell-1}+1} c_{n,j}^2\right)^{1/2}. \end{aligned}$$

Consequently by stationarity,

$$\begin{aligned} &\mathbb{E}\Big(\Big|(S_{n,m-2^{\ell-1}} - S_{n,m-2^{\ell}}) - (T_{n,m-2^{\ell-1}+1} - T_{n,m-2^{\ell}+1})\Big|^{p-2}\Big|\mathbb{E}\big(\xi_m^2 - 1|\mathcal{F}_{m-2^{\ell-1}}\big)\Big|\Big) \\ &\quad \le \|(S_{n,m-2^{\ell-1}} - S_{n,m-2^{\ell}}) - (T_{n,m-2^{\ell-1}+1} - T_{n,m-2^{\ell}+1})\|_p^{p-2}\|\mathbb{E}\big(\xi_{2^{\ell-1}}^2 - 1|\mathcal{F}_0\big)\|_{p/2} \\ &\quad \le C\left(\sum_{j=m-2^{\ell}+1}^{m-2^{\ell-1}+1} c_{n,j}^2\right)^{(p-2)/2} \|\mathbb{E}\big(\xi_{2^{\ell-1}}^2 - 1|\mathcal{F}_0\big)\|_{p/2}. \end{aligned}$$

Setting

$$B(m,k) := \Big(\max_{j\in\mathbb{Z}} c_{n,j}^2 + \sum_{j=m-k+2}^{N} c_{n,j}^2\Big)^{(p-r)/2}, \tag{6.15}$$

it follows that for any $\alpha \in [0, p-2]$,

$$
\begin{aligned}
&\sum_{m=1}^{N} c_{n,m}^2 \sum_{\ell=1}^{[\log_2 m]} \mathbb{E}\Big(\big(f''_{N-(m-2^{\ell-1})-1,n}(S_{n,m-2^{\ell-1}}) - f''_{N-(m-2^{\ell})-1,n}(S_{n,m-2^{\ell}})\big)(\xi_m^2 - 1)\Big) \\
&\quad \le C \sum_{m=1}^{N} c_{n,m}^2 \sum_{\ell=1}^{[\log_2 m]} \frac{\|\mathbb{E}(\xi_{2^{\ell-1}}^2 - 1|\mathcal{F}_0)\|_{p/2}}{B(m, 2^{\ell-1})} \left(\sum_{j=m-2^{\ell}+1}^{m-2^{\ell-1}+1} c_{n,j}^2\right)^{(p-2)/2} \\
&\quad \le C \max_{j\in\mathbb{Z}} |c_{n,j}|^{\alpha} \sum_{\ell=0}^{[\log_2 N]-1} 2^{\alpha\ell/2} \|\mathbb{E}(\xi_{2^{\ell}}^2 - 1|\mathcal{F}_0)\|_{p/2} \\
&\qquad \times \sum_{m=2^{\ell}+1}^{N} \frac{c_{n,m}^2}{B(m, 2^{\ell})} \left(\sum_{j=m-2^{\ell+1}+1}^{m-2^{\ell}+1} c_{n,j}^2\right)^{(p-2-\alpha)/2} .
\end{aligned}
$$

Applying Hölder's inequality we get that

$$
\begin{aligned}
&\sum_{m=2^{\ell}+1}^{N} \frac{c_{n,m}^2}{B(m, 2^{\ell})} \left(\sum_{j=m-2^{\ell+1}+1}^{m-2^{\ell}+1} c_{n,j}^2\right)^{(p-2-\alpha)/2} \\
&\quad \le \left(\sum_{m=2^{\ell}+1}^{N} \frac{|c_{n,m}|^{p-\alpha}}{B(m, 2^{\ell})}\right)^{2/(p-\alpha)} \\
&\qquad \times \left(\sum_{m=2^{\ell}+1}^{N} \frac{\left(\sum_{j=m-2^{\ell+1}+1}^{m-2^{\ell}+1} c_{n,j}^2\right)^{(p-\alpha)/2}}{B(m, 2^{\ell})}\right)^{(p-\alpha-2)/(p-\alpha)} \\
&\quad \le \left(\sum_{m=2^{\ell}+1}^{N} \frac{|c_{n,m}|^{p-\alpha}}{B(m, 1)}\right)^{2/(p-\alpha)} \\
&\qquad \times \left((2^{\ell}+1)^{(p-\alpha-2)/2} \sum_{m=2^{\ell}+1}^{N} \frac{\sum_{j=m-2^{\ell+1}+1}^{m-2^{\ell}+1} |c_{n,j}|^{p-\alpha}}{B(m, 2^{\ell})}\right)^{(p-\alpha-2)/(p-\alpha)} .
\end{aligned}
$$

Since $\sum_{j=m-2^{\ell+1}+2}^{m-2^{\ell}+1} c_{n,j}^2 \le 2^{\ell} \max_{j\in\mathbb{Z}} c_{n,j}^2$, we clearly have

$$
(2^{\ell}+1)\left(\max_{j\in\mathbb{Z}} c_{n,j}^2 + \sum_{j=m-2^{\ell}+2}^{N} c_{n,j}^2\right) \ge \max_{j\in\mathbb{Z}} c_{n,j}^2 + \sum_{j=m-2^{\ell+1}+2}^{N} c_{n,j}^2 .
$$

Hence

$$\begin{aligned}\sum_{m=2^{\ell+1}}^{N} \frac{\sum_{j=m-2^{\ell+1}+1}^{m-2^{\ell}+1} |c_{n,j}|^{p-\alpha}}{B(m,2^{\ell})} &\leq (2^{\ell}+1)^{(p-r)/2} \sum_{m=2^{\ell+1}}^{N} \frac{\sum_{j=m-2^{\ell+1}+1}^{m-2^{\ell}+1} |c_{n,j}|^{p-\alpha}}{B(m,2^{\ell+1})} \\ &\leq \sum_{j=1}^{N} (2^{\ell}+1)^{(p-r)/2} |c_{n,j}|^{p-\alpha} \sum_{m=1}^{N} \frac{\mathbb{1}_{j+2^{\ell}-1 \leq m \leq j+2^{\ell+1}-1}}{B(m,2^{\ell+1})} \\ &\leq (2^{\ell}+1)^{(p-r+2)/2} \sum_{j=1}^{N} \frac{|c_{n,j}|^{p-\alpha}}{B(j,1)}.\end{aligned}$$

Taking into account all the above considerations we derive that for any $\alpha \in [0, p-2]$,

$$\begin{aligned}&\sum_{m=1}^{N} c_{n,m}^2 \sum_{\ell=1}^{[\log_2 m]} \mathbb{E}\Big(\big(f''_{N-(m-2^{\ell-1})-1,n}(S_{n,m-2^{\ell-1}}) - f''_{N-(m-2^{\ell})-1,n}(S_{n,m-2^{\ell}})\big)(\xi_m^2-1)\Big) \\ &\quad \leq C \sum_{\ell \geq 0} 2^{\ell\beta} \|\mathbb{E}\big(\xi_{2^\ell}^2 - 1|\mathcal{F}_0\big)\|_{p/2} K_{p,r,\alpha}(n,N).\end{aligned}$$

Since $(\|\mathbb{E}\big(\xi_m^2 - 1|\mathcal{F}_0\big)\|_{p/2})_{m\geq 0}$ is a decreasing sequence, Condition (3.1) implies that

$$\sum_{\ell=0}^{\infty} 2^{\ell\beta} \|\mathbb{E}\big(\xi_{2^\ell}^2 - 1|\mathcal{F}_0\big)\|_{p/2} < \infty. \tag{6.16}$$

Hence

$$\begin{aligned}&\sum_{m=1}^{N} c_{n,m}^2 \sum_{\ell=1}^{[\log_2 m]} \mathbb{E}\Big(\big(f''_{N-(m-2^{\ell-1})-1,n}(S_{n,m-2^{\ell-1}}) - f''_{N-(m-2^{\ell})-1,n}(S_{n,m-2^{\ell}})\big)(\xi_m^2-1)\Big) \\ &\quad \leq C K_{p,r,\alpha}(n,N).\end{aligned} \tag{6.17}$$

It remains to upper bound

$$\sum_{m=2}^{N} c_{n,m}^2 \mathbb{E}\Big(f''_{N-(m-2^{[\log_2 m]})-1,n}(S_{n,m-2^{[\log_2 m]}})(\xi_m^2-1)\Big).$$

We first use the inequality (6.14), and the fact that $\mathbb{E}\big(f''_{N-(m-2^{[\log_2 m]})-1,n}(0)(\xi_m^2-1)\big) = 0$. Using the notation (6.15), we get

$$\begin{aligned}&\sum_{m=2}^{N} c_{n,m}^2 \mathbb{E}\Big(f''_{N-(m-2^{[\log_2 m]})-1,n}(S_{n,m-2^{[\log_2 m]}})(\xi_m^2-1)\Big) \\ &\quad \leq C \sum_{m=2}^{N} \frac{c_{n,m}^2}{B(m,2^{[\log_2 m]})} \|S_{n,m-2^{[\log_2 m]}}\|_p^{p-2} \|\mathbb{E}(\xi_m^2|\mathcal{F}_{m-2^{[\log_2 m]}}) - 1\|_{p/2}.\end{aligned}$$

Now we notice that $2^{[\log_2 m]} \geq m/2$. Consequently

$$\|\mathbb{E}(\xi_m^2|\mathcal{F}_{m-2^{[\log_2 m]}}) - 1\|_{p/2} \le \|\mathbb{E}(\xi_{[m/2]}^2|\mathcal{F}_0) - 1\|_{p/2},$$

and by the Burkholder inequality,

$$\|S_{n,m-2^{[\log_2 m]}}\|_p^{p-2} \le C\|\xi_0\|_p^{p-2} \left(\sum_{j=1}^{m-2^{[\log_2 m]}} c_{n,j}^2\right)^{(p-2)/2}.$$

Consequently, since $\alpha \in [0, p-2]$,

$$\sum_{m=2}^{N} c_{n,m}^2 \mathbb{E}\Big(f''_{N-(m-2^{[\log_2 m]})-1,n}(S_{n,m-2^{[\log_2 m]}})(\xi_m^2 - 1)\Big)$$
$$\le C \max_{j\in\mathbb{Z}} |c_{n,j}|^\alpha \sum_{m=2}^{N} m^{\alpha/2} \frac{c_{n,m}^2}{B(m, 2^{[\log_2 m]})} \left(\sum_{j=1}^{m-2^{[\log_2 m]}} c_{n,j}^2\right)^{(p-2-\alpha)/2}$$
$$\times \|\mathbb{E}(\xi_{[m/2]}^2|\mathcal{F}_0) - 1\|_{p/2}.$$

If $\alpha = p-2$, using the fact that $m^{\alpha/2}\|\mathbb{E}(\xi_{[m/2]}^2|\mathcal{F}_0) - 1\|_{p/2} = O(1)$, we get the following bound:

$$\sum_{m=2}^{N} c_{n,m}^2 \mathbb{E}\Big(f''_{N-(m-2^{[\log_2 m]})-1,n}(S_{n,m-2^{[\log_2 m]}})(\xi_m^2 - 1)\Big) \le C \max_{j\in\mathbb{Z}} |c_{n,j}|^{p-2} \sum_{m=2}^{N} \frac{c_{n,m}^2}{B(m, 1)}.$$

Now in the case where $\alpha \in [0, p-2[$, using Hölder's inequality we then derive that

$$\sum_{m=2}^{N} c_{n,m}^2 \mathbb{E}\Big(f''_{N-(m-2^{[\log_2 m]})-1,n}(S_{n,m-2^{[\log_2 m]}})(\xi_m^2 - 1)\Big)$$
$$\le C \max_{j\in\mathbb{Z}} |c_{n,j}|^\alpha \left(\sum_{m=2}^{N} \frac{|c_{n,m}|^{p-\alpha}}{B(m, 1)}\right)^{2/(p-\alpha)}$$
$$\times \left(\sum_{m=2}^{N} m^{\frac{\alpha(p-\alpha)}{2(p-\alpha-2)} + \frac{p-\alpha-2}{2}} \|\mathbb{E}(\xi_{[m/2]}^2|\mathcal{F}_0) - 1\|_{p/2}^{(p-\alpha)/(p-\alpha-2)} \right.$$
$$\left. \times \frac{\sum_{j=1}^{m-2^{[\log_2 m]}} |c_{n,j}|^{p-\alpha}}{B(m, 2^{[\log_2 m]})}\right)^{(p-\alpha-2)/(p-\alpha)}.$$

Since for any $j = 1, \ldots, m - 2^{[\log_2 m]}$, $\sum_{k=j+1}^{m-2^{[\log_2 m]}+1} c_{n,k}^2 \le m \max_{k\in\mathbb{Z}} c_{n,k}^2$, we get that

$$(m+1)^{(p-r)/2} B(m, 2^{[\log_2 m]}) \ge B(j, 1).$$

Hence,

$$\sum_{m=2}^{N} m^{\frac{\alpha(p-\alpha)}{2(p-\alpha-2)}+\frac{p-\alpha-2}{2}} \|\mathbb{E}(\xi_{[m/2]}^2|\mathcal{F}_0)-1\|_{p/2}^{(p-\alpha)/(p-\alpha-2)} \frac{\sum_{j=1}^{m-2^{[\log_2 m]}} |c_{n,j}|^{p-\alpha}}{B(m,2^{[\log_2 m]})}$$
$$\leq 2^{(p-r)/2} \sum_{m=2}^{N} \frac{m^{\beta(p-\alpha)/(p-\alpha-2)}}{m} \|\mathbb{E}(\xi_{[m/2]}^2|\mathcal{F}_0)-1\|_{p/2}^{(p-\alpha)/(p-\alpha-2)} \sum_{j\geq 1} \frac{|c_{n,j}|^{p-\alpha}}{B(j,1)}.$$

Consequently

$$\sum_{m=2}^{N} c_{n,m}^2 \mathbb{E}\Big(f''_{N-(m-2^{[\log_2 m]})-1,n}(S_{n,m-2^{[\log_2 m]}})(\xi_m^2-1)\Big) \leq C K_{p,r,\alpha}(n,N), \tag{6.18}$$

provided that

$$\sum_{m=2}^{N} \frac{m^{\beta(p-\alpha)/(p-\alpha-2)}}{m} \|\mathbb{E}(\xi_{[m/2]}^2|\mathcal{F}_0)-1\|_{p/2}^{(p-\alpha)/(p-\alpha-2)} < \infty,$$

which holds under (3.1). From (6.7), (6.8), (6.13), (6.17) and (6.18), we conclude that (6.2) holds.

6.3. Proof of Lemma 3.1

For any $m \in \mathbb{Z}$, we set

$$t_{n,m} = c_{n,m}/\max_{j\in\mathbb{Z}} |c_{n,j}|. \tag{6.19}$$

With this notation we then derive that

$$\widetilde{L}_{p,r,\alpha}(n) = \max_{j\in\mathbb{Z}} |c_{n,j}|^r \sum_{m\in\mathbb{Z}} \frac{|t_{n,m}|^{p-\alpha}}{1+\sum_{j=m+1}^{\infty} t_{n,j}^2} \Bigg(1+\sum_{j=m+1}^{\infty} t_{n,j}^2\Bigg)^{1-(p-r)/2}.$$

Now since $|t_{n,m}| \leq 1$ and $p-\alpha \geq 2$,

$$\widetilde{L}_{p,r,\alpha}(n) \leq \max_{j\in\mathbb{Z}} |c_{n,j}|^r \sum_{m\in\mathbb{Z}} \frac{|t_{n,m}|^{p-\alpha}}{1+\sum_{j=m+1}^{\infty} |t_{n,j}|^{p-\alpha}} \Bigg(1+\sum_{j=m+1}^{\infty} |t_{n,j}|^{p-\alpha}\Bigg)^{1-(p-r)/2}. \tag{6.20}$$

Using again the fact that $|t_{n,m}| \leq 1$,

$$\frac{|t_{n,m}|^{p-\alpha}}{1+\sum_{j=m+1}^{\infty} |t_{n,j}|^{p-\alpha}} \leq \frac{2|t_{n,m}|^{p-\alpha}}{1+|t_{n,m}|^{p-\alpha}+2\sum_{j=m+1}^{\infty} |t_{n,j}|^{p-\alpha}}.$$

Now, for any $m \in \mathbb{Z}$, set

$$u_{n,m} = 1+2\sum_{j=m+1}^{\infty} |t_{n,j}|^{p-\alpha}, \tag{6.21}$$

and notice that

$$\frac{2|t_{n,m}|^{p-\alpha}}{1+|t_{n,m}|^{p-\alpha}+2\sum_{j=m+1}^{\infty}|t_{n,j}|^{p-\alpha}}=\frac{2(u_{n,m-1}-u_{n,m})}{(u_{n,m-1}-u_{n,m})+2u_{n,m}}.$$

Applying the inequality: $\log(1+x)\geq 2x/(x+2)$ for $x>0$, to $x=(u_{n,m-1}-u_{n,m})/u_{n,m}$, we then derive that

$$\frac{|t_{n,m}|^{p-\alpha}}{1+\sum_{j=m+1}^{\infty}|t_{n,j}|^{p-\alpha}}\leq \log\left(\frac{u_{n,m-1}}{u_{n,m}}\right).$$

In addition, we notice that for any $r\in[p-2,p]$,

$$\left(1+\sum_{j=m+1}^{\infty}|t_{n,j}|^{p-\alpha}\right)^{1-(p-r)/2}\leq x^{1-(p-r)/2}\quad\text{for any } x\geq u_{n,m}.$$

It follows that

$$\begin{aligned}\sum_{m\in\mathbb{Z}}\frac{|t_{n,m}|^{p-\alpha}}{\left(1+2\sum_{j=m+1}^{\infty}|t_{n,j}|^{p-\alpha}\right)^{(p-r)/2}}&\leq\sum_{m\in\mathbb{Z}}\int_{u_{n,m}}^{u_{n,m-1}}x^{-(p-r)/2}\mathrm{d}x\\&=\int_{1}^{1+2\sum_{j\in\mathbb{Z}}|t_{n,j}|^{p-\alpha}}x^{-(p-r)/2}\mathrm{d}x.\end{aligned}$$

Hence,

$$\sum_{m\in\mathbb{Z}}\frac{|t_{n,m}|^{p-\alpha}}{\left(1+2\sum_{j=m+1}^{\infty}|t_{n,j}|^{p-\alpha}\right)^{(p-r)/2}}\leq\begin{cases}\dfrac{2}{2-p+r}\left(1+2\sum_{j\in\mathbb{Z}}|t_{n,j}|^{p-\alpha}\right)^{(2-p+r)/2} & \text{if } r\in]p-2,p]\\ \log\left(1+2\sum_{j\in\mathbb{Z}}|t_{n,j}|^{p-\alpha}\right) & \text{if } r=p-2,\end{cases}\tag{6.22}$$

which gives the result by taking into account (6.20) and (6.19).

6.4. *Proof of Theorem 3.2*

As in the proof of Theorem 3.1 and with the same notation, it suffices to prove that for any positive integer N,

$$\zeta_r(P_{S_{n,N}},G_{v_{n,N}^2\sigma^2})\leq C\big(M_n^r+\widehat{L}_{p,r}(n,N)\big),\tag{6.23}$$

where

$$\widehat{L}_{p,r}(n,N)=\sum_{k=1}^{N}\frac{|c_{n,k}|^p}{\left(M_n^2+\sum_{j=k+1}^{N}c_{n,j}^2\right)^{(p-r)/2}}.$$

We modify the proof of Theorem 3.1 as follows: here Z_n is a $\mathcal{N}(0,M_n^2)$-distributed random variable independent of $(\xi_i)_{i\in\mathbb{Z}}$ and $(Y_i)_{i\in\mathbb{Z}}$. It follows that (6.12) is replaced by

$$|f_{N-m,n}|_{\Lambda_p}\le C\left(M_n^2+\sum_{j=m+1}^{N}c_{n,j}^2\right)^{(r-p)/2}. \tag{6.24}$$

We then follow the lines of the proof of Theorem 3.1 to get the bound (6.13) for $\sum_{m=1}^N R_m$ except that we replace $K_{p,r,\alpha}(n,N)$ by $\widehat{L}_{p,r}(n,N)$. In addition we upper bound $D'=\sum_{m=1}^N D'_m$, where D'_m is defined by (6.10), in a different way. We write that for any $m=1,\ldots,N$,

$$\begin{aligned} f''_{N-m,n}(S_{n,m-1}) \;=\; & f''_{N-1,n}(0)+\sum_{j=1}^{m-1}\big(f''_{N-(m-j)-1,n}(S_{n,m-j}) \\ & -f''_{N-(m-j),n}(S_{n,m-j-1})\big), \end{aligned} \tag{6.25}$$

since $S_{n,0}=0$. Now for any $j=1,\ldots,m-1$,

$$\begin{aligned} &\mathbb{E}\Big(\big(f''_{N-(m-j)-1,n}(S_{n,m-j})-f''_{N-(m-j),n}(S_{n,m-j-1})\big)(\xi_m^2-1)\Big)\\ &\quad=\mathbb{E}\Big(\big(f''_{N-(m-j)-1,n}(S_{n,m-j})-f''_{N-(m-j)-1,n}(S_{n,m-j-1}+T_{n,m-j+1}-T_{n,m-j})\big)\\ &\qquad\times\mathbb{E}\big(\xi_m^2-1|\mathcal{F}_{m-j}\big)\Big). \end{aligned}$$

Using (6.14) (with M_n instead of $\max_{j\in\mathbb{Z}}|c_{n,j}|$), the stationarity of $(\xi_i)_{i\in\mathbb{Z}}$, and the fact that $\mathbb{E}(f''_{N-1,n}(0)(\xi_m^2-1))=0$, it follows that

$$\begin{aligned} D' &=\sum_{m=1}^{N}c_{n,m}^2\sum_{j=1}^{m-1}\mathbb{E}\left(\left(f''_{N-(m-j)-1,n}(S_{n,m-j})-f''_{N-(m-j),n}(S_{n,m-j-1})\right)(\xi_m^2-1)\right)\\ &\le C\sum_{j=1}^{N-1}\||\xi_0|^{p-2}\mathbb{E}\big(\xi_j^2-1|\mathcal{F}_0\big)\|_1\\ &\quad\times\sum_{m=j+1}^{N}c_{n,m}^2|c_{n,m-j}|^{p-2}\left(M_n^2+\sum_{k=m-j+2}^{N}c_{n,k}^2\right)^{(r-p)/2}\\ &\quad+C\sum_{j=1}^{N-1}\|\mathbb{E}\big(\xi_j^2-1|\mathcal{F}_0\big)\|_1\\ &\quad\times\sum_{m=j+1}^{N}c_{n,m}^2|c_{n,m-j+1}|^{p-2}\left(M_n^2+\sum_{k=m-j+2}^{N}c_{n,k}^2\right)^{(r-p)/2}. \end{aligned}$$

From Hölder's inequality, we get that

$$\sum_{m=j+1}^{N} c_{n,m}^2 |c_{n,m-j}|^{p-2} \left(M_n^2 + \sum_{k=m-j+2}^{N} c_{n,k}^2 \right)^{\frac{r-p}{2}} \le \sum_{m=1}^{N} \frac{|c_{n,m}|^p}{\left(M_n^2 + \sum_{k=m+2}^{N} c_{n,k}^2 \right)^{(p-r)/2}}.$$

Similarly

$$\sum_{m=j+1}^{N} c_{n,m}^2 |c_{n,m-j+1}|^{p-2} \left(M_n^2 + \sum_{k=m-j+2}^{N} c_{n,k}^2 \right)^{\frac{r-p}{2}} \le \sum_{m=1}^{N} \frac{|c_{n,m}|^p}{\left(M_n^2 + \sum_{k=m+2}^{N} c_{n,k}^2 \right)^{(p-r)/2}}.$$

Consequently if (3.9) holds,

$$D' \le C \sum_{m=1}^{N} \frac{|c_{n,m}|^p}{\left(M_n^2 + \sum_{k=m+2}^{N} c_{n,k}^2 \right)^{(p-r)/2}}. \tag{6.26}$$

This ends the proof of the theorem.

6.5. *Proof of Comment 3.4*

Using the notation and arguments given at the beginning of the proof of Theorem 3.1, it suffices to prove that for any positive integer N,

$$\zeta_r(P_{S_{n,N}}, G_{v_{n,N}^2 \sigma^2}) \le C \max_{j \in \mathbb{Z}} |c_{n,j}|^{p-2}. \tag{6.27}$$

With this aim, we follow the lines of the proof of Theorem 3.2 with $M_n = \max_{j \in \mathbb{Z}} |c_{n,j}|$, except that we give more precise upper bounds for the terms $\sum_{m=1}^{N} R_m$ and $D' = \sum_{m=1}^{N} D'_m$ than (6.13) and (6.26) (recall that R_m and D'_m are defined respectively in (6.11) and (6.10)). Indeed Taylor's formula at orders 2 and 3 and the strict stationarity give

$$\begin{aligned} R_m \le{} & c_{n,m}^2 \mathbb{E}\left(\xi_0^2 \left(\| f''_{N-m,n} \|_\infty \wedge \frac{1}{6} \| f^{(3)}_{N-m,n} \|_\infty \max_{j \in \mathbf{Z}} |c_{n,j}| |\xi_0| \right) \right) \\ & + \frac{|c_{n,m}|^3}{6} \| f^{(3)}_{N-m,n} \|_\infty \mathbb{E}(|Y_0|^3). \end{aligned} \tag{6.28}$$

In addition using the fact that $S_{n,0} = 0$ and $\mathbb{E}(f''_{N-m,n}(0)(\xi_m^2 - 1)) = 0$ for every $m = 1, \ldots, N$, and the stationarity, we derive that

$$D'_m = c_{n,m}^2 \sum_{j=1}^{m-1} \mathbb{E}\big((f''_{N-m,n}(S_{n,m-j}) - f''_{N-m,n}(S_{n,m-j-1}))(\xi_m^2 - 1) \big).$$

Then using again the stationarity, we get

$$D' = \sum_{m=1}^{N} D'_m \le C \sum_{j=1}^{N-1} \sum_{m=j+1}^{N} c_{n,m}^2 \mathbb{E}\big(A_{N,m}(\xi_0, Y_1) \big| \mathbb{E}(\xi_j^2 - 1 | \mathcal{F}_0) \big| \big), \tag{6.29}$$

where $A_{N,m}(\xi_0, Y_1) := \|f''_{N-m,n}\|_\infty \wedge \big(\|f^{(3)}_{N-m,n}\|_\infty \max_k |c_{n,k}|(|\xi_0| + |Y_1|)\big)$. Notice now that for any positive integer i, $\|f^{(i)}_{N-m,n}\|_\infty = \|f * \phi^{(i)}_{\delta_n}\|_\infty$ where ϕ_t is the density of the law $N(0, t^2)$ and $\delta_n^2 = \max_{j\in\mathbb{Z}} |c_{n,j}|^2 + \sum_{j=m+1}^N c_{n,j}^2$. Since f belongs to Λ_r (i.e. $|f|_{\Lambda_r} \le 1$) and $r = p-2$, it follows from Remark 6.1 of Dedecker et al. [4] that for any integer $i \ge 2$,

$$\|f^{(i)}_{N-m,n}\|_\infty \le C_i \left(\sum_{j=m+1}^N c_{n,j}^2 + \max_{j\in\mathbb{Z}} |c_{n,j}|^2\right)^{(p-2-i)/2}, \tag{6.30}$$

where C_i is a positive constant depending on i.

We first upper bound D'. Starting from (6.29), using (6.30) and the notation (6.19), and setting for any $m = 0, \dots, N$,

$$u_{n,m,N} = 1 + 2\sum_{j=m+1}^N t_{n,j}^2, \tag{6.31}$$

we obtain that for any $j = 1, \dots, N-1$,

$$\begin{aligned}&\sum_{m=j+1}^N c_{n,m}^2 \mathbb{E}\big(A_{N,m}(\xi_0, Y_1)\big|\mathbb{E}(\xi_j^2 - 1|\mathcal{F}_0)\big|\big)\\ &\quad\le C \max_{k\in\mathbb{Z}} |c_{n,k}|^{p-2}\\ &\qquad\times \mathbb{E}\left(\big|\mathbb{E}(\xi_j^2 - 1|\mathcal{F}_0)\big| \sum_{m=j+1}^N \frac{t_{n,m}^2}{\left(1 + \sum_{k=m+1}^N t_{n,k}^2\right)^{(4-p)/2}} B_{N,m}(\xi_0, Y_1)\right),\end{aligned} \tag{6.32}$$

where $B_{N,m}(\xi_0, Y_1) := 1 \wedge \Big((1 + \sum_{k=m+1}^N t_{n,k}^2)^{-1/2}(|\xi_0| + |Y_1|)\Big)$. Now, we upper bound $B_{N,m}(\xi_0, Y_1)$ as follows:

$$\begin{aligned}B_{N,m}(\xi_0, Y_1) &\le \mathbb{1}_{1+\sum_{k=m+1}^N t_{n,k}^2 \le (|\xi_0|+|Y_1|)^2}\\ &\quad + \frac{|\xi_0| + |Y_1|}{\left(1 + \sum_{k=m+1}^N t_{n,k}^2\right)^{1/2}} \mathbb{1}_{1+\sum_{k=m+1}^N t_{n,k}^2 \ge (|\xi_0|+|Y_1|)^2}.\end{aligned}$$

Since $t_{n,m}^2 \le 1$, $u_{n,m-1,N} \le 3(1 + \sum_{k=m+1}^N t_{n,k}^2)$. Moreover $u_{n,m,N} \ge 1 + \sum_{k=m+1}^N t_{n,k}^2$. Therefore,

$$B_{N,m}(\xi_0, Y_1) \le \mathbb{1}_{u_{n,m-1,N} \le 3(|\xi_0|+|Y_1|)^2} + \frac{|\xi_0| + |Y_1|}{\left(1 + \sum_{k=m+1}^N t_{n,k}^2\right)^{1/2}} \mathbb{1}_{u_{n,m,N} \ge (|\xi_0|+|Y_1|)^2}. \tag{6.33}$$

As in the proof of Lemma 3.1, using the fact that for all integers m, $t_{n,m}^2 \le 1$, the following inequality is valid:

$$\frac{t_{n,m}^2}{1+\sum_{j=m+1}^{N} t_{n,j}^2} \le \log\left(\frac{u_{n,m-1,N}}{u_{n,m,N}}\right). \tag{6.34}$$

In addition since $p > 2$, for any $x \ge u_{n,m,N}$,

$$\left(1+\sum_{j=m+1}^{N} |t_{n,j}|^2\right)^{1-(4-p)/2} \le u_{n,m,N}^{1-(4-p)/2} \le x^{1-(4-p)/2}.$$

Consequently using the two above inequalities, since $p > 2$,

$$\begin{aligned}
&\sum_{m=j+1}^{N} \frac{t_{n,m}^2}{\left(1+\sum_{k=m+1}^{N} t_{n,k}^2\right)^{(4-p)/2}} \mathbb{1}_{u_{n,m-1,N} \le 3(|\xi_0|+|Y_1|)^2} \\
&\le \sum_{m=j+1}^{N} \left(\int_{u_{n,m,N}}^{u_{n,m-1,N}} x^{-(4-p)/2} \mathrm{d}x\right) \mathbb{1}_{u_{n,m-1,N} \le 3(|\xi_0|+|Y_1|)^2} \\
&\le \int_{1}^{3(|\xi_0|+|Y_1|)^2} x^{-(4-p)/2} \mathrm{d}x \le 2(p-2)^{-1} 3^{(p-2)/2} (|\xi_0|+|Y_1|)^{p-2}.
\end{aligned} \tag{6.35}$$

On the other hand, using once again the fact that $u_{n,m-1,N} \le 3(1+\sum_{k=m+1}^{N} t_{n,k}^2)$ and that $p < 3$, we get that

$$\begin{aligned}
&\sum_{m=j+1}^{N} \frac{t_{n,m}^2}{\left(1+\sum_{k=m+1}^{N} t_{n,k}^2\right)^{(5-p)/2}} \mathbb{1}_{u_{n,m,N} \ge (|\xi_0|+|Y_1|)^2} \\
&\le 3^{(3-p)/2} \sum_{m=j+1}^{N} \frac{t_{n,m}^2}{1+\sum_{k=m+1}^{N} t_{n,k}^2} u_{n,m-1,N}^{(p-3)/2} \mathbb{1}_{u_{n,m,N} \ge (|\xi_0|+|Y_1|)^2}.
\end{aligned}$$

Consequently by using (6.34) and the fact that if $x \le u_{n,m-1,N}$ then $u_{n,m-1,N}^{(p-3)/2} \le x^{(p-3)/2}$ (since $p < 3$), we derive that

$$\begin{aligned}
&\sum_{m=j+1}^{N} \frac{t_{n,m}^2}{\left(1+\sum_{k=m+1}^{N} t_{n,k}^2\right)^{(5-p)/2}} \mathbb{1}_{u_{n,m,N} \ge (|\xi_0|+|Y_1|)^2} \\
&\le 3^{(3-p)/2} \sum_{m=j+1}^{N-1} \left(\int_{u_{n,m,N}}^{u_{n,m-1,N}} x^{(p-3)/2-1} \mathrm{d}x\right) \mathbb{1}_{u_{n,m,N} \ge (|\xi_0|+|Y_1|)^2} \\
&\le 3^{(3-p)/2} \int_{(|\xi_0|+|Y_1|)^2}^{\infty} x^{-(5-p)/2} \mathrm{d}x \le 2(3-p)^{-1} 3^{(3-p)/2} (|\xi_0|+|Y_1|)^{p-3}.
\end{aligned} \tag{6.36}$$

Starting from (6.32) and considering the bounds (6.33), (6.35) and (6.36), we then derive that

$$D' \le C \max_{j\in\mathbb{Z}} |c_{n,j}|^{p-2} \sum_{j=1}^{N-1} \|(|\xi_0|^{p-2} \vee 1)|\mathbb{E}(\xi_j^2|\mathcal{F}_0) - \sigma^2|\|_1.$$

Consequently, under (3.9),

$$D' \le C \max_{j\in\mathbb{Z}} |c_{n,j}|^{p-2}. \tag{6.37}$$

Now, we upper bound $\sum_{m=1}^N R_m$. According to the arguments developed above, we first get that

$$\begin{aligned}
&\sum_{m=1}^N c_{n,m}^2 \mathbb{E}\Big(\xi_0^2\big(\|f''_{N-m,n}\|_\infty \wedge \|f^{(3)}_{N-m,n}\|_\infty \max_{j\in\mathbf{Z}} |c_{n,j}||\xi_0|\big)\Big)\\
&\quad\le C \max_{j\in\mathbb{Z}} |c_{n,j}|^{p-2} \mathbb{E}\left(\xi_0^2 \sum_{m=1}^N \frac{t_{n,m}^2}{\left(1+\sum_{k=m+1}^N t_{n,k}^2\right)^{(4-p)/2}} \mathbb{1}_{u_{n,m-1,N}\le 3\xi_0^2}\right)\\
&\qquad + C \max_{j\in\mathbb{Z}} |c_{n,j}|^{p-2} \mathbb{E}\left(|\xi_0|^3 \sum_{m=1}^N \frac{t_{n,m}^2}{\left(1+\sum_{k=m+1}^N t_{n,k}^2\right)^{(5-p)/2}} \mathbb{1}_{u_{n,m,N}\ge \xi_0^2}\right).
\end{aligned}$$

With the same arguments as were used to get (6.35) and (6.36), we obtain that

$$\begin{aligned}
&\sum_{m=1}^N c_{n,m}^2 \mathbb{E}\Big(\xi_0^2\big(\|f''_{N-m,n}\|_\infty \wedge \|f^{(3)}_{N-m,n}\|_\infty \max_{j\in\mathbf{Z}} |c_{n,j}||\xi_0|\big)\Big)\\
&\quad\le C \max_{j\in\mathbb{Z}} |c_{n,j}|^{p-2} \|\xi_0\|_p^p.
\end{aligned} \tag{6.38}$$

On the other hand, considering the bound (6.30), we get that

$$\sum_{m=1}^N |c_{n,m}|^3 \|f^{(3)}_{N-m,n}\|_\infty \le C \max_{j\in\mathbb{Z}} |c_{n,j}|^{p-2} \sum_{m=1}^N \frac{t_{n,m}^2}{\left(1+\sum_{k=m+1}^N t_{n,k}^2\right)^{(5-p)/2}}.$$

As for getting (6.36), we then derive that

$$\begin{aligned}
\sum_{m=1}^N |c_{n,m}|^3 \|f^{(3)}_{N-m,n}\|_\infty &\le C \max_{j\in\mathbb{Z}} |c_{n,j}|^{p-2} 3^{(3-p)/2} \int_1^\infty x^{-(5-p)/2} \mathrm{d}x\\
&\le C \max_{j\in\mathbb{Z}} |c_{n,j}|^{p-2}.
\end{aligned} \tag{6.39}$$

Starting from (6.28) and collecting the bounds (6.38) and (6.39), we obtain that

$$\sum_{m=1}^{N} R_m \le C \max_{j\in\mathbb{Z}} |c_{n,j}|^{p-2}. \tag{6.40}$$

Taking into account the bounds (6.37) and (6.40), (6.27) is proven, and so is Comment 3.4.

6.6. *Proof of Comment 4.5*

We shall prove that (4.11) implies (3.5) for $\mathcal{F}_0 = \sigma(\eta_i, i \le 0)$. Notice that

$$\mathbb{E}(\xi_n^2|\mathcal{F}_0) - \mathbb{E}(\xi_0^2) = \mathbb{E}(\sigma_n^2|\mathcal{F}_0) - \mathbb{E}(\sigma_n^2),$$

where σ_n^2 is defined in (4.10). Since $\mathbb{E}(\eta_0^2) = 1$ and $\sum_{j\ge1} c_j < 1$, the unique stationary solution to (4.10) is given by Giraitis et al. [6]:

$$\sigma_n^2 = c + c\sum_{\ell=1}^{\infty} \sum_{j_1,\dots,j_\ell=1}^{\infty} c_{j_1}\dots c_{j_\ell}\eta_{n-j_1}^2\dots\eta_{n-(j_1+\cdots+j_\ell)}^2. \tag{6.41}$$

Setting $\kappa = \|\eta_0\|_p^2 \sum_{j\ge1} c_j$, it follows that

$$\begin{aligned}
&\|\mathbb{E}(\xi_n^2|\mathcal{F}_0) - \mathbb{E}(\xi_0^2)\|_{p/2} \\
&\quad \le 2c \left\| \sum_{\ell=1}^{\infty} \sum_{j_1,\dots,j_\ell=1}^{\infty} c_{j_1}\dots c_{j_\ell}\eta_{n-j_1}^2\dots\eta_{n-(j_1+\cdots+j_\ell)}^2 \mathbb{1}_{j_1+\cdots+j_\ell\ge n} \right\|_{p/2} \\
&\quad \le 2c \sum_{\ell=1}^{\infty} \sum_{j_1,\dots,j_\ell=1}^{\infty} \sum_{k=1}^{\ell} c_{j_1}\dots c_{j_\ell} \mathbb{1}_{j_k\ge[n/\ell]} \|\eta_0\|_p^{2\ell} \\
&\quad \le 2c\|\eta_0\|_p^2 \sum_{\ell=1}^{\infty} \ell\kappa^{\ell-1} \sum_{k=[n/\ell]}^{\infty} c_k.
\end{aligned}$$

Consequently under (4.11), $\|\mathbb{E}(\xi_n^2|\mathcal{F}_0) - \mathbb{E}(\xi_0^2)\|_{p/2} = O(n^{-b})$, so (3.5) holds as soon as $b > p/2 - 1$.

6.7. *Proof of Theorem 4.1*

Following Volný [16], if (4.12) holds, then

$$\xi_0 = d_0 + Z - Z \circ T, \tag{6.42}$$

where Z belongs to $\mathbb{L}^p$. For any $j \ge 1$, let $R_j = \sum_{k=1}^{j}(\xi_k - d_k)$ and $\tilde{R}_j = \sum_{k=1}^{j}(\xi_{-k} - d_{-k})$, and let also $R_0 = \tilde{R}_0 = 0$. From (6.42), we easily infer that

$$\|R_j\|_p \le 2\|Z\|_p \quad \text{and} \quad \|\tilde{R}_j\|_p \le 2\|Z\|_p. \tag{6.43}$$

Let $T_n = \sum_{j\in\mathbb{Z}} c_{n,j} d_j$, and $\Delta_n = S_n - T_n$. For any $n \ge 1$, one has that

$$\Delta_n = c_{n,0}(\xi_0 - d_0) + \sum_{j=1}^{\infty} c_{n,j}(\xi_j - d_j) + \sum_{j=1}^{\infty} c_{n,-j}(\xi_{-j} - d_{-j}).$$

By assumption (4.13), for any $n \ge 1$, $\sum_{j\in\mathbb{Z}} |c_{n,j} - c_{n,j-1}| < \infty$. From (6.43), it follows that the two series

$$\sum_{j=1}^{\infty}(c_{n,j-1}-c_{n,j})R_{j-1} \quad \text{and} \quad \sum_{j=1}^{\infty}(c_{n,-j-1}-c_{n,-j})\tilde{R}_{j-1}$$

converge in $\mathbb{L}^p$. Hence, an Abel transformation gives

$$\Delta_n = c_{n,0}(\xi_0-d_0)+\sum_{j=1}^{\infty}(c_{n,j-1}-c_{n,j})R_{j-1}+\sum_{j=1}^{\infty}(c_{n,-j-1}-c_{n,-j})\tilde{R}_{j-1},$$

and so

$$\|\Delta_n\|_p \le 2\|Z\|_p\left(|c_{n,0}|+\sum_{j\in\mathbb{Z}}|a_{n+j}-a_j|\right). \tag{6.44}$$

On the other hand, for $i>0$,

$$|c_{n,i}| = \left|c_{n,0}+\sum_{j=1}^{i}(c_{n,j}-c_{n,j-1})\right| \le |c_{n,0}|+\sum_{j\in\mathbb{Z}}|a_{n+j}-a_j|,$$

and the same upper bound is valid for $c_{n,i}$ with $i<0$. It follows that Condition (4.13) implies Condition (4.4). Consequently, if the sequence $(d_i)_{i\in\mathbb{Z}}$ satisfies (3.5), it follows from Corollary 4.1 that

$$\zeta_r(P_{T_n/v_n}, G_{\sigma^2}) \le \begin{cases} Cn^{1-p/2} & \text{if } r\in]p-2,p] \\ Cn^{1-p/2}\log n & \text{if } r=p-2, \end{cases} \tag{6.45}$$

where $\sigma^2=\mathbb{E}(d_0^2)=\sum_{k\in\mathbb{Z}}\mathbb{E}(\xi_0\xi_k)$.

We now complete the proof of Theorem 4.1 with the help of (6.44) and (6.45).

If $f\in\Lambda_r$ with $r\in[p-2,1]$, then

$$|\mathbb{E}(f(v_n^{-1}S_n)-f(v_n^{-1}T_n))| \le v_n^{-r}\|\Delta_n\|_p^r \le Cv_n^{-r}\left(|c_{n,0}|+\sum_{j\in\mathbb{Z}}|a_{n+j}-a_j|\right)^r,$$

and the last bound is $O(n^{-r/2})$ by (4.13). Then if $r\in[p-2,1]$, Items 1 and 2 follow from (6.45).

If $f\in\Lambda_r$ with $r\in]1,2]$, from the proof of Lemma 5.2 of Dedecker et al. [4], we get that

$$|\mathbb{E}(f(v_n^{-1}S_n)-f(v_n^{-1}T_n))| \le v_n^{-r}(\|\Delta_n\|_r\|T_n\|_r^{r-1}+\|\Delta_n\|_r^r).$$

Since $\|T_n\|_r\le\|T_n\|_2=\sigma v_n$, we obtain that

$$|\mathbb{E}(f(v_n^{-1}S_n)-f(v_n^{-1}T_n))| \le C\left(\frac{|c_{n,0}|+\sum_{j\in\mathbb{Z}}|a_{n+j}-a_j|}{v_n}+\frac{\left(|c_{n,0}|+\sum_{j\in\mathbb{Z}}|a_{n+j}-a_j|\right)^r}{v_n^r}\right),$$

and the last bound is $O(n^{-1/2})$ by (4.13). Then if $r\in]1,2]$, Item 2 follows from (6.45).

We turn now to the proof of Item 3. If $f \in \Lambda_r$ with $r \in]2, p]$ and if $\sigma > 0$, we set $\alpha_n = \|S_n\|_2 \|T_n\|_2^{-1}$. Following the proof of Lemma 5.2 of Dedecker et al. [4] and setting $\tilde{\Delta}_n = \Delta_n + (1-\alpha_n)T_n$, we get that

$$\mathbb{E}(f(v_n^{-1}S_n) - f(\alpha_n v_n^{-1}T_n)) \le \frac{1}{(r-1)v_n^r}\alpha_n^{r-1}\|\tilde{\Delta}_n\|_r \|T_n\|_r^{r-1} + \alpha_n^{r-2}\|\tilde{\Delta}_n\|_r^2 \frac{\|T_n\|_r^{r-2}}{2v_n^r} + \frac{\|\tilde{\Delta}_n\|_r^r}{2v_n^r}.$$

Now $\alpha_n = O(1)$ and $\|\tilde{\Delta}_n\|_r \le \|\Delta_n\|_r + |1-\alpha_n|\|T_n\|_r$. Since $|\|S_n\|_2 - \|T_n\|_2| \le \|\Delta_n\|_2$, we infer by using (6.44) (with p replaced by r) that

$$|1-\alpha_n| \le C\frac{|c_{n,0}| + \sum_{j\in\mathbb{Z}} |a_{n+j} - a_j|}{v_n}.$$

Hence, applying Burkholder's inequality for martingales, we infer that

$$\|\tilde{\Delta}_n\|_r \le C\left(|c_{n,0}| + \sum_{j\in\mathbb{Z}} |a_{n+j} - a_j|\right).$$

Consequently

$$\mathbb{E}(f(v_n^{-1}S_n) - f(\alpha_n v_n^{-1}T_n)) \le C\left(\frac{|c_{n,0}| + \sum_{j\in\mathbb{Z}} |a_{n+j} - a_j|}{v_n} + \frac{\left(|c_{n,0}| + \sum_{j\in\mathbb{Z}} |a_{n+j} - a_j|\right)^2}{v_n^2} + \frac{\left(|c_{n,0}| + \sum_{j\in\mathbb{Z}} |a_{n+j} - a_j|\right)^r}{v_n^r}\right),$$

and the last bound is $O(n^{-1/2})$ by (4.13). Then if $r \in]2, p]$ and $\sigma^2 > 0$, Item 3 follows from (6.45) and the fact that

$$\zeta_r(P_{\alpha_n v_n^{-1}T_n}, G_{\sigma_n^2}) = \alpha_n^r \zeta_r(P_{v_n^{-1}T_n}, G_{\sigma^2}).$$

It remains to consider the case where $r \in]2, p]$ and $\sigma^2 = 0$. In this case $S_n = \Delta_n$. Let Y be a $\mathcal{N}(0, 1)$ random variable. Following the proof of Lemma 5.2 of Dedecker et al. [4], we get that for any $f \in \Lambda_r$,

$$|\mathbb{E}(f(v_n^{-1}S_n) - f(\sigma_n Y))| \le \frac{1}{(r-1)v_n}\|\bar{\Delta}_n\|_r \|\sigma_n Y\|_r^{(r-1)} + \frac{1}{2v_n^2}\|\bar{\Delta}_n\|_r^2 \|\sigma_n Y\|_r^{r-2} + \frac{1}{2v_n^r}\|\bar{\Delta}_n\|_r^r,$$

where $\bar{\Delta}_n = \Delta_n - \sigma_n v_n Y$. Since $\sigma_n v_n = \|\Delta_n\|_2 \le \|\Delta_n\|_r$ and since $\|\Delta_n\|_r = O(v_n n^{-1/2})$ by (6.44) and Condition (4.13), we get that $\|\bar{\Delta}_n\|_r = O(v_n n^{-1/2})$. The result follows. □

6.8. Proof of Corollary 5.2

We apply Theorem 3.2 with $M_n = \alpha_n (\sum_{i=1}^n \alpha_i^2)^{-1} = O(n^{\gamma-1})$ which gives the upper bound $\zeta_r(P_{S_n/v_n}, G_{\sigma^2}) \le C v_n^{-r}(M_n^r + \widehat{L}_{p,r}^{(1)}(n) + \widehat{L}_{p,r}^{(2)}(n))$ where

$$v_n^{-r}\widehat{L}_{p,r}^{(1)}(n) := v_n^r \sum_{k=1}^{[n/2]-1} \frac{\alpha_k^p}{\left(\sum_{j=k+1}^n \alpha_j^2\right)^{(p-r)/2}}.$$

and

$$v_n^{-r}\widehat{L}_{p,r}^{(2)}(n) := v_n^r \sum_{k=[n/2]}^{n} \frac{\alpha_k^p}{\left(\alpha_n^2 + \sum_{j=k+1}^n \alpha_j^2\right)^{(p-r)/2}}.$$

With the above choice of M_n, we get that for any $r \in [p-2, p]$,

$$v_n^{-r} M_n^r \le C n^{1-p/2}. \tag{6.46}$$

Now

$$v_n^{-r}\widehat{L}_{p,r}^{(1)}(n) \le C v_n^r \big(n\alpha_n^2\big)^{(r-p)/2} \sum_{k=1}^{[n/2]-1} \alpha_k^p.$$

Therefore,

$$v_n^{-r}\widehat{L}_{p,r}^{(1)}(n) \le \begin{cases} Cn^{(2\gamma-1)p/2} & \text{if } \gamma p > 1 \\ Cn^{1-p/2}\log(n) & \text{if } \gamma p = 1 \\ Cn^{1-p/2} & \text{if } \gamma p < 1. \end{cases} \tag{6.47}$$

Now, we upper bound $v_n^{-r}\widehat{L}_{p,r}^{(2)}(n)$ by noticing first that

$$v_n^{-r}\widehat{L}_{p,r}^{(2)}(n) \le C v_n^r \alpha_{[n/2]}^p \alpha_n^{r-p} \sum_{k=1}^{[n/2]-1} k^{(r-p)/2},$$

which leads to

$$v_n^{-r}\widehat{L}_{p,r}^{(2)}(n) \le \begin{cases} Cn^{1-p/2} & \text{if } r \in]p-2, p] \\ Cn^{1-p/2}\log(n) & \text{if } r = p-2. \end{cases} \tag{6.48}$$

Collecting the bounds (6.46)–(6.48), we obtain (5.3) in the case where $r \in]p-2, p]$ and (5.4) in the case where $r = p-2$.

Acknowledgement

We thank the referee for his or her careful reading of the manuscript and for numerous comments, which helped to improve the first version of this paper, and in particular the statement of Corollary 5.2.

References

[1] J. Beran, Statistics for long-memory processes, in: Monographs on Statistics and Applied Probability, vol. 61, Chapman and Hall, New York, 1994.
[2] A. Bikelis, Estimates of the remainder term in the central limit theorem, Litovsk. Mat. Sb. 6 (1966) 323–346.
[3] E. Bolthausen, Exact convergence rates in some martingale central limit theorems, Ann. Probab. 10 (1982) 672–688.
[4] J. Dedecker, F. Merlevède, E. Rio, Rates of convergence for minimal distances in the central limit theorem under projective criteria, Electron. J. Probab. 14 (2009) 978–1011.
[5] R.M. Dudley, Real Analysis and Probability, in: Chapman & Hall Mathematics Series, Wadsworth, Inc. Chapman & Hall, New York, London, 1989.
[6] L. Giraitis, P. Kokoszka, R. Leipus, Stationary ARCH models: dependence structure and central limit theorem, Econometric Theory 16 (2000) 3–22.
[7] E.J. Hannan, Multiple Time Series, John Wiley and Sons, Inc., New York, London, Sydney, 1970.
[8] E.J. Hannan, The central limit theorem for time series regression, Stoch. Process. Appl. 9 (1979) 281–289.
[9] E.J. Hannan, C.C. Heyde, On limit theorems for quadratic functions of discrete time series, Ann. Math. Statist. 43 (1972) 2058–2066.
[10] C.C. Heyde, On the central limit theorem and iterated logarithm law for stationary processes, Bull. Aust. Math. Soc. 12 (1975) 1–8.
[11] C. Jan, Vitesse de convergence dans le TCL pour des processus associés à des systèmes dynamiques et aux produits de matrices aléatoires. Thèse, université Rennes 1, 2001.
[12] M. Peligrad, S. Utev, Central limit theorem for linear processes, Ann. Probab. 25 (1997) 443–456.
[13] E. Rio, Distances minimales et distances idéales, C. R. Acad. Sci., Paris I 326 (1998) 1127–1130.
[14] P.M. Robinson, Large-sample inference for nonparametric regression with dependent errors, Ann. Statist. 25 (1997) 2054–2083.
[15] A.I. Sakhanenko, Estimates in an invariance principle, in: Limit Theorems of Probability Theory, "Nauka" Sibirsk. Otdel., Novosibirsk. Tr. Inst. Mat. 5 (1985) 27–44, 175.
[16] D. Volný, Approximating martingales and the central limit theorem for strictly stationary processes, Stoch. Process. Appl. 44 (1993) 41–74.
[17] W.B. Wu, Strong invariance principles for dependent random variables, Ann. Probab. 35 (2007) 2294–2320.